AMBIGUOUS CLASS NUMBER FORMULAS

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Abstract. An elementary proof of Chevalley's ambiguous class number formula is presented.

1. Introduction

In Gras' book [2, p. 178, p. 180] one finds Chevalley's ambiguous class formulas. In Lemmermeyer [3] one finds a modern and elementary proof. This Note gives a different elementary proof of this result, which uses basic results proved in Lang's book [1].

Let K/k be a cyclic extension of number fields with Galois group $G = \mathrm{Gal}(K/k) = \langle\sigma\rangle$, where σ is a generator of G. Denote by $\mathfrak{o}$ and $\mathfrak{O}$ the ring of integers of k and K, respectively. Let ∞ and ∞_r (resp. $\widetilde{\infty}$ and $\widetilde{\infty}_r$) denote the set of infinite and real places of k (resp. of K), respectively, and $\mathbb{A}_k$ (resp. $\mathbb{A}_K$) the adele ring of k (resp. K). We shall identify a real cycle $\mathfrak{c}$ with its support, which is a subset of real places. Let $r_k : \widetilde{\infty} \to \infty$ denote the restriction to k.

Let $\widetilde{\mathfrak{c}}$ be a real cycle on K which is stable under the G-action. Denote by

$$\mathrm{Cl}(K, \widetilde{\mathfrak{c}}) := \frac{\mathbb{A}_K^\times}{K^\times \widehat{\mathfrak{O}}^\times K_\infty(\widetilde{\mathfrak{c}})^\times} \tag{1.1}$$

the narrow ideal class group of K with respect to $\widetilde{\mathfrak{c}}$, where $\widehat{\mathfrak{O}}$ is the profinite completion of $\mathfrak{O}$, and $K_\infty(\widetilde{\mathfrak{c}})^\times = \{a = (a_w) \in K_\infty^\times \mid a_w > 0 \quad \forall\, w \in \widetilde{\mathfrak{c}}\}$. Similarly one defines $\mathrm{Cl}(k, \mathfrak{c})$ for any real cycle $\mathfrak{c}$ on k. The group G acts on the finite abelian group $\mathrm{Cl}(K, \widetilde{\mathfrak{c}})$. Its G-invariant subgroup $\mathrm{Cl}(K, \widetilde{\mathfrak{c}})^G$ is called the *ambiguous ideal class group* (with respect to $\widetilde{\mathfrak{c}}$).

Let $\mathfrak{c}$ be the real cycle on k such that $\infty_r - \mathfrak{c} = r_k(\widetilde{\infty}_r - \widetilde{\mathfrak{c}})$ and $\mathfrak{c}_0 := r_k(\widetilde{\mathfrak{c}})$. One has $\mathfrak{c} = \mathfrak{c}_0 \infty_r^c$, where ∞_r^c is the set of real places of k which does not split completely in K. Let $N_{K/k}$ denote the norm map from K to k. The cycle $\mathfrak{c}$ is determined by the property $N_{K/k}(K_\infty(\widetilde{\mathfrak{c}})^\times) = k_\infty(\mathfrak{c})^\times$. Put $\mathfrak{o}(\mathfrak{c})^\times := \mathfrak{o}^\times \cap i_\infty^{-1}(k_\infty(\mathfrak{c})^\times)$, where $i_\infty : k^\times \to k_\infty^\times$ is the diagonal embedding. Denote by V_f the set of finite places of k. Let $e(v)$ denote the ramification index of any place w over $v \in V_f$.

Theorem 1.1. *One has*

$$\# \mathrm{Cl}(K, \widetilde{\mathfrak{c}})^G = \frac{\# \mathrm{Cl}(k, \mathfrak{c}) \prod_{v \in V_f} e(v)}{[K : k][\mathfrak{o}(\mathfrak{c})^\times : \mathfrak{o}(\mathfrak{c})^\times \cap N_{K/k}(K^\times)]}. \tag{1.2}$$

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When $\widetilde{\mathfrak{c}} = \widetilde{\infty}_r$, we get the restricted version of the formula stated in [2, p. 178]. When $\widetilde{\mathfrak{c}} = \emptyset$, using an elementary fact

$$\# \operatorname{Cl}(k, \infty_r^c) = \frac{h(k) \cdot 2^{|\infty_r^c|}}{[\mathfrak{o}^\times : \mathfrak{o}(\infty_r^c)^\times]},$$

we get the ordinary version of the formula stated in [2, p. 180].

2. Proof of Theorem 1.1

Define the norm ideal class group $N(K, \widetilde{\mathfrak{c}})$ by

$$N(K, \widetilde{\mathfrak{c}}) := \frac{N_{K/k}(\mathbb{A}_K^\times)}{N_{K/k}(K^\times \widehat{\mathfrak{O}}^\times K_\infty(\widetilde{\mathfrak{c}})^\times)}. \tag{2.1}$$

Consider the commutative diagram of two short exact sequences (by Hilbert's Theorem 90)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{A}_K^{\times 1-\sigma} \cap U & \longrightarrow & U & \xrightarrow{N_{K/k}} & N_{K/k}(U) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{A}_K^{\times 1-\sigma} & \longrightarrow & \mathbb{A}_K^\times & \xrightarrow{N_{K/k}} & N_{K/k}(\mathbb{A}_K^\times) & \longrightarrow & 1, \end{array} \tag{2.2}$$

where $U = K^\times \widehat{\mathfrak{O}}^\times K_\infty(\widetilde{\mathfrak{c}})^\times$. The snake lemma gives the short exact sequence

$$1 \longrightarrow \operatorname{Cl}(K, \widetilde{\mathfrak{c}})^{1-\sigma} \longrightarrow \operatorname{Cl}(K, \widetilde{\mathfrak{c}}) \longrightarrow N(K, \widetilde{\mathfrak{c}}) \longrightarrow 1 \tag{2.3}$$

as one has an isomorphism $\mathbb{A}_K^{\times 1-\sigma}/(\mathbb{A}_K^{\times 1-\sigma} \cap U) \simeq \operatorname{Cl}(K, \widetilde{\mathfrak{c}})^{1-\sigma}$. On the other hand we have the short exact sequence

$$1 \longrightarrow \operatorname{Cl}(K, \widetilde{\mathfrak{c}})^G \longrightarrow \operatorname{Cl}(K, \widetilde{\mathfrak{c}}) \longrightarrow \operatorname{Cl}(K, \widetilde{\mathfrak{c}})^{1-\sigma} \longrightarrow 1, \tag{2.4}$$

which with (2.3) shows the following result.

Lemma 2.1. *We have* $\# \operatorname{Cl}(K, \widetilde{\mathfrak{c}})^G = \# N(K, \widetilde{\mathfrak{c}})$.

Define

$$\operatorname{Cl}(k, \mathfrak{c}, \mathfrak{O}) := \frac{\mathbb{A}_k^\times}{k^\times k_\infty(\mathfrak{c})^\times N_{K/k}(\widehat{\mathfrak{O}}^\times)}.$$

Lemma 2.2. *The group* $N(K, \widetilde{\mathfrak{c}})$ *is isomorphic to a subgroup* $H \subset \operatorname{Cl}(k, \mathfrak{c}, \mathfrak{O})$ *of index* $[K : k]$.

Proof. Put $A := N_{K/k}(\mathbb{A}_K^\times)$, $B := N_{K/k}(K^\times \widehat{\mathfrak{O}}^\times K_\infty(\widetilde{\mathfrak{c}})^\times)$, $C := k^\times$ and $H := CA/CB$. The group H is a subgroup in $\operatorname{Cl}(k, \mathfrak{c}, \mathfrak{O})$, which is of index $[K : k]$ by the global norm index theorem [1, p. 193]. One has $A \cap C = N_{K/k}(K^\times) \subset B$ by the Hasse norm theorem [1, p. 195]. The lemma follows from

$$N(K, \widetilde{\mathfrak{c}}) = A/B = A/(A \cap C)B \simeq CA/CB = H. \blacksquare$$

Consider the exact sequence

$$1 \longrightarrow \frac{\mathfrak{o}(\mathfrak{c})^\times}{\mathfrak{o}(\mathfrak{c})^\times \cap N(\widehat{\mathfrak{O}}^\times)} \longrightarrow \frac{\widehat{\mathfrak{o}}^\times}{N(\widehat{\mathfrak{O}}^\times)} \longrightarrow \operatorname{Cl}(k, \mathfrak{c}, \mathfrak{O}) \longrightarrow \operatorname{Cl}(k, \mathfrak{c}) \longrightarrow 1. \tag{2.5}$$

It is easy to see $\mathfrak{o}(\mathfrak{c})^\times \cap N_{K/k}(\widehat{\mathfrak{O}}^\times) = \mathfrak{o}(\mathfrak{c})^\times \cap N_{K/k}(K^\times)$ from the Hasse norm theorem. The local norm index theorem [1, p. 188, Lemma 4] gives

$$\#\left(\frac{\widehat{\mathfrak{o}}^\times}{N(\widehat{\mathfrak{O}}^\times)}\right) = \prod_{v\in V_f} e(v). \tag{2.6}$$

Combining Lemma 2.2, (2.5) and (2.6) we get

$$\# N(K,\widetilde{\mathfrak{c}}) = \frac{\#\,\mathrm{Cl}(k,\mathfrak{c},\mathfrak{O})}{[K:k]} = \frac{\#\,\mathrm{Cl}(k,\mathfrak{c})\prod_{v\in V_f} e(v)}{[K:k][\mathfrak{o}(\mathfrak{c})^\times : \mathfrak{o}(\mathfrak{c})^\times \cap N_{K/k}(K^\times)]}. \tag{2.7}$$

Theorem 1.1 follows from Lemma 2.1 and (2.7). ∎

Remark 2.3. We do not know whether $\mathrm{Cl}(K,\widetilde{\mathfrak{c}})^G$ and $N(K,\widetilde{\mathfrak{c}})$ are isomorphic as abelian groups or whether there is a natural bijection between them. When $[K:k]=2$ and $\#\,\mathrm{Cl}(K,\widetilde{\mathfrak{c}})^{1-\sigma}$ is odd, we show that there is a natural isomorphism

$$N(K,\widetilde{\mathfrak{c}}) \simeq \mathrm{Cl}(K,\widetilde{\mathfrak{c}})^G. \tag{2.8}$$

The map $1-\sigma : \mathrm{Cl}(K,\widetilde{\mathfrak{c}}) \to \mathrm{Cl}(K,\widetilde{\mathfrak{c}})^{1-\sigma}$ restricted to $\mathrm{Cl}(K,\widetilde{\mathfrak{c}})^{1-\sigma}$ is the squared map Sq, which is an isomorphism from our assumption. The inverse of Sq defines a section of (2.4), and hence an isomorphism $\mathrm{Cl}(K,\widetilde{\mathfrak{c}}) \simeq \mathrm{Cl}(K,\widetilde{\mathfrak{c}})^G \oplus \mathrm{Cl}(K,\widetilde{\mathfrak{c}})^{1-\sigma}$. The assertion (2.8) then follows.

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