Characterization of Besov Spaces Associated with Parabolic Sections

Meifang Cheng
Chin-Cheng Lin
Meng Qu

NCTS/Math Technical Report
2016-010

National Center for
Theoretical Sciences
Mathematics Division, Taiwan

CHARACTERIZATION OF BESOV SPACES ASSOCIATED WITH PARABOLIC SECTIONS

MEIFANG CHENG*, CHIN-CHENG LIN†, AND MENG QU‡

Abstract. We study the Besov spaces associated with a family of parabolic sections which are closely related to the parabolic Monge-Ampère equation. We demonstrate their duals and an embedding theorem for these Besov spaces.

1. Introduction

In 1996, Caffarelli and Gutiérrez [1] studied real variable theory related to the Monge-Ampère equation. They considered a family of convex sets in $\mathbb{R}^n$, $\mathcal{F} = \{S(x,t) : x \in \mathbb{R}^n, t > 0\}$, satisfying certain axioms of affine invariance, and a Borel measure satisfying a doubling condition with respect to the family $\mathcal{F}$. They developed a Besicovitch-type covering lemma for the family $\mathcal{F}$ and used this covering lemma with the doubling property of the Borel measure mentioned above to set up a variant of the Calderón-Zygmund decomposition in terms of the members of $\mathcal{F}$. Let $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ be a strictly convex smooth function and consider the Monge-Ampère measure $\mu := \det D^2\phi$ generated by ϕ, where $D^2\phi$ denotes the Hessian matrix of ϕ. For a given function u, it is easy to see that

$$\det D^2(\phi + tu) = \det D^2\phi + t\,\mathrm{tr}(\Phi\, D^2 u) + \ldots + t^n \det D^2 u,$$

where Φ is the matrix of the cofactors of $D^2\phi$ and $\mathrm{tr}(A)$ means the trace of the matrix A. For $x \in \mathbb{R}^n$ and $t > 0$, Caffarelli and Gutiérrez [1] introduced a family of *elliptic sections* associated with ϕ by

$$S_\phi(x,t) = \{y \in \mathbb{R}^n : \phi(y) - \phi(x) - \nabla\phi(x)\cdot(y-x) < t\}.$$

These sets play crucial role in the study of Monge-Ampère equation and the linearized Monge-Ampère equation $L_\phi u = \mathrm{tr}(\Phi\, D^2 u)$ (cf. [2]). In [1], Caffarelli and Gutiérrez generalized $S_\phi(x,t)$ to an abstract family of convex sets $S(x,t)$ satisfying properties (A), (B), and (C) given in [1, page 1078], and we call these $S(x,t)$ to be *generalized elliptic sections*. The elliptic sections $S_\phi(x,t)$ associated with ϕ is an example of generalized elliptic sections.

2010 *Mathematics Subject Classification.* 42B35.

Key words and phrases. Besov spaces, Calderón reproducing formula, embedding theorem, Monge-Ampère equation, parabolic sections.

* Supported by NNSF of China (No. 11201003).

† Supported by MOST of Taiwan under Grant #MOST 103-2115-M-008-003-MY3.

‡ Supported by NNSF of China (No. 11471033), NSF of Anhui Provincial (No. 1408085MA01), and University NSR Project of Anhui Province (No. KJ2014A087).

In 1999, Huang [11] pondered the Harnack inequality for nonnegative solutions of the linearized parabolic Monge-Ampère equation

$$u_t - \mathrm{tr}((D^2\phi(x))^{-1}D^2u) = 0, \tag{1.1}$$

where $u_t = \frac{\partial u}{\partial t}$, $(D^2\phi(x))^{-1}$ is the inverse matrix of $D^2\phi(x)$, and ϕ is a strictly convex smooth function defined on $\mathbb{R}^n$ such that $\det D^2\phi\, dxdt$ satisfies a certain doubling condition on the parabolic sections $S(x,r) \times (t - c_1 r, t + c_2 r]$ associated with ϕ and is uniformly absolutely continuous with respect to Lebesgue measure. More precisely, it was assumed in [11] that the Monge-Ampère measure $\mu = \det D^2\phi$ satisfies the following doubling property in terms of sections:

$$\mu(S(x,t)) \le C\mu\left(\frac{1}{2}S(x,t)\right) \qquad \text{for all } S(x,t), \tag{1.2}$$

where $C > 0$ and $\frac{1}{2}S(x,t)$ denotes $\frac{1}{2}$-dilation of $S(x,t)$ with respect to its center of mass. It was also required in [11] that μ satisfies a stronger uniform continuity condition: for any given $\delta_1 \in (0,1)$, there exists $\delta_2 \in (0,1)$ such that, for any sections S and any measurable subset $E \subset S$,

$$\frac{|E|}{|S|} < \delta_2 \quad \text{implies} \quad \frac{\mu(E)}{\mu(S)} < \delta_1. \tag{1.3}$$

We note that (1.3) implies (1.2). Also, Huang obtained a Besicovitch-type covering lemma with respect to parabolic sections. Then he considered the parabolic Monge-Ampère measure $\mathcal{M}$ generated by $\phi(x) - t$, i.e., $d\mathcal{M} = \det D^2\phi\, dxdt$, and obtained a variant of the Calderón-Zygmund decomposition in terms of parabolic sections and $\mathcal{M}$ under the uniform continuity condition on μ. Using such a Calderón-Zygmund decomposition, Huang showed an invariant Harnack's inequality on parabolic sections as follows.

Theorem 1.1. *Let u be a nonnegative classical solution of* (1.1) *in* $S(x_0, \bar\theta R) \times (t_0 - \frac{3}{2}R, t_0 + 2R]$, *where $\bar\theta$ is a large geometric constant. Then*

$$\sup_{Q^-} u \le C \inf_{Q^+} u,$$

where $Q^+ = S(x_0, R) \times (t_0 + R, t_0 + 2R]$ *and* $Q^- = S(x_0, R) \times (t_0 - R, t_0]$.

Parabolic sections also appeared in the work of Gutiérrez and Huang [8], where they proved the $W^{2,p}$ estimates for the parabolic Monge-Ampère equation

$$-u_t \det D^2 u = f, \qquad (x,t) \in \Omega \times (0,T) \subset \mathbb{R}^n \times \mathbb{R}, \tag{1.4}$$

with some suitable conditions on f and Ω being a bounded convex set. Initially (1.4) was introduced by Krylov [12] in 1976. Its connection with maximum principles for parabolic equations was observed by Krylov, and was developed further by Tso [20] and Nazarov and Ural'tseva [15]. Equation (1.4) also arose in the work of Tso [19] on the Gauss curvature flow of convex hypersurfaces. The first initial-boundary value problem for (1.4) was studied by R. H. Wang and G. L. Wang [21, 22]. Moreover, Daskalopoulos and Savin [7] obtained a $C^{1,\alpha}$ estimate for the following parabolic Monge-Ampère equation

$$u_t = b(x,t)(\det D^2 u)^p, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where $p > 0$ and $b(x,t)$ is a bounded positive measurable function. Recently, Tang [17] investigated interior estimates of solutions to (1.4) in the case that f satisfies VMO-type condition, and such VMO spaces are defined in terms of parabolic sections. It is our hope that the spaces studied in the current paper provide another direction in the investigation of the regularity of solutions to parabolic Monge-Ampère equation with initial data in Besov spaces.

We first recall the definition of (generalized) parabolic sections. Suppose that $\varphi : [0,\infty) \mapsto [0,\infty)$ is a monotonic increasing function satisfying

$$\varphi(0) = 0, \qquad \lim_{r\to\infty} \varphi(r) = \infty, \qquad \varphi(2r) \le C\varphi(r),$$

where C is a constant depending on φ only. Define the *generalized parabolic sections*, which will be called parabolic sections below for simplicity, by

$$Q_\varphi(z,r) = S(x,r) \times \Big(t - \frac{\varphi(r)}{2}, t + \frac{\varphi(r)}{2}\Big),$$

where $z = (x,t) \in \mathbb{R}^n \times \mathbb{R}$, $r > 0$, and $S(x,r)$ is the generalized elliptic sections. Note that this definition reduces to the one given in [11] by choosing $\varphi(r) = r$. We will work for a fixed φ satisfying the above description through the paper, and hence use $Q(z,r)$ to express $Q_\varphi(z,r)$ for simplicity. An affine transformation $\widetilde{T}$ on $\mathbb{R}^{n+1}$ is said to *normalize* $Q(z_0,r)$ if

$$K\Big(0, \frac{1}{n}\Big) \subset \widetilde{T}\big(Q(z_0,r)\big) \subset K(0,1),$$

where $K(z,r) = B(x,r) \times \big(t - \frac{r^2}{2}, t + \frac{r^2}{2}\big)$, $\widetilde{T}(x,t) := (Tx, \frac{t-t_0}{\varphi(r)})$, and T is an affine transformation on $\mathbb{R}^n$ normalizing $S(x_0,r)$; that is,

$$B\Big(0, \frac{1}{n}\Big) \subset T\big(S(x_0,r)\big) \subset B(0,1).$$

Here we use $B(x,r)$ to denote the ball in $\mathbb{R}^n$ centered at x and with radius r. Note that the restriction of $\widetilde{T}$ to t-axis maps $\big(t_0 - \frac{\varphi(r)}{2}, t_0 + \frac{\varphi(r)}{2}\big)$ onto $(-\frac{1}{2}, \frac{1}{2})$. The family

$$\mathcal{P} = \{Q(z,r) : z = (x,t) \in \mathbb{R}^n \times \mathbb{R},\ r > 0\}$$

of parabolic sections satisfies the following properties (see [11, page 2029]).

(**A**) There exist positive constants K_1, K_2, K_3 and ε_1, ε_2 such that, given two parabolic sections $Q(z_0,r_0)$, $Q(z,r)$ in $\mathcal{P}$ with $r \le r_0$ and an affine transformation $\widetilde{T}$ that normalizes $Q(z_0,r_0)$, if

$$Q(z_0,r_0) \ \cap \ Q(z,r) \ \neq \ \varnothing,$$

then there exists $z' = (x',t') \in K(0,K_3)$, depending only on both $Q(z_0,r_0)$ and $Q(z,r)$, satisfying

$$B\left(x', K_2\Big(\frac{r}{r_0}\Big)^{\varepsilon_2}\right) \times \left(t' - \frac{1}{2}\frac{\varphi(r)}{\varphi(r_0)}, t' + \frac{1}{2}\frac{\varphi(r)}{\varphi(r_0)}\right) \subset \widetilde{T}\big(Q(z,r)\big)$$

$$\subset B\left(x', K_1\Big(\frac{r}{r_0}\Big)^{\varepsilon_1}\right) \times \left(t' - \frac{1}{2}\frac{\varphi(r)}{\varphi(r_0)}, t' + \frac{1}{2}\frac{\varphi(r)}{\varphi(r_0)}\right)$$

and

$$\widetilde{T}(z) = (Tx, t') \in B\left(x', \frac{1}{2}K_2\Big(\frac{r}{r_0}\Big)^{\varepsilon_2}\right) \times \{t'\}.$$

(**B**) There exists $\iota > 0$ such that, for any parabolic section $Q(z_0, r) \in \mathcal{P}$ and $z \notin Q(z_0, r)$, if $\widetilde{T}$ is an affine transformation that normalizes $Q(z_0, r)$, then

$$K\big(\widetilde{T}(z), \epsilon^{\iota}\big) \cap \widetilde{T}\big(Q(z_0, (1-\epsilon)r)\big) = \varnothing \qquad \text{for} \quad 0 < \epsilon < 1.$$

(**C**) $\bigcap_{r>0} Q(z, r) = \{z\}$ and $\bigcup_{r>0} Q(z, r) = \mathbb{R}^{n+1}$.

In addition, we also assume that a Borel measure ν is given, which is finite on compact sets, no point mass, $\nu(\mathbb{R}^{n+1}) = \infty$, and satisfies the following *doubling property* with respect to $\mathcal{P}$; that is, there exists a constant C_ν such that

$$\nu\big(Q(z, 2r)\big) \le C_\nu \nu\big(Q(z, r)\big), \qquad \forall\, Q(z, r) \in \mathcal{P}. \tag{1.5}$$

We note that the parabolic Monge-Ampère measure $\mathcal{M}$ using in [11] satisfies (1.5).

Since the parabolic sections are similar to elliptic cylinders, by properties (**A**) and (**B**) of parabolic sections, it is easy to obtain the following *engulfing property.* There exists a constant $\theta \ge 1$, depending only on ι, K_1, and ε_1, such that for each $z' \in Q(z, r) \in \mathcal{P}$ we have

$$Q(z, r) \subset Q(z', \theta r) \qquad \text{and} \qquad Q(z', r) \subset Q(z, \theta r). \tag{1.6}$$

Define a quasi-metric d on $\mathbb{R}^{n+1}$ with respect to $\mathcal{P}$ by

$$d(z, w) = \inf\{r : z \in Q(w, r) \text{ and } w \in Q(z, r)\},$$

which satisfies the triangle inequality

$$d(z, w) \le \theta\big(d(z, u) + d(u, w)\big) \qquad \text{for any } z, u, w \in \mathbb{R}^{n+1}. \tag{1.7}$$

Also,

$$Q\Big(z, \frac{r}{2\theta}\Big) \subset B_d(z, r) \subset Q(z, r) \qquad \text{for any } z \in \mathbb{R}^{n+1} \text{ and } r > 0, \tag{1.8}$$

where $B_d(z, r) := \{w \in \mathbb{R}^{n+1} : d(z, w) < r\}$ denotes the d-ball centered at z with radius r. By (1.5) and (1.8), if we choose $k_0 \in \mathbb{N}$ satisfying $2^{k_0 - 2} \ge \theta$, then

$$\nu(B_d(z, 2r)) \le C_\nu^{k_0} \nu(B_d(z, r)) \qquad \text{for any } z \in \mathbb{R}^{n+1} \text{ and } r > 0.$$

Hence, $(\mathbb{R}^{n+1}, d, \nu)$ is a space of homogeneous type introduced by Coifman and Weiss [4]. Macías and Segovia [14, Theorems 2] have shown that one can replace d by another quasi-metric ρ such that there exist constants $c > 1$ and $\varepsilon \in (0, 1)$ satisfying

$$\begin{cases} c^{-1} d(z, w) \le \rho(z, w) \le c d(z, w) \qquad \text{for } z, w \in \mathbb{R}^{n+1}; \\ |\rho(z, w) - \rho(z', w)| \le c(\rho(z, z'))^{\varepsilon}[\rho(z, w) + \rho(z', w)]^{1-\varepsilon} \qquad \text{for } z, z', w \in \mathbb{R}^{n+1}. \end{cases} \tag{1.9}$$

By (1.7) and (1.9), it is easy to check that ρ satisfies the triangle inequality

$$\rho(z, w) \le A\big(\rho(z, u) + \rho(u, w)\big) \qquad \text{for any } z, w, u \in \mathbb{R}^{n+1}, \tag{1.10}$$

where $A = c^2\theta$. Through the paper, we always assume that the quasi-metric ρ satisfies the regularity condition (1.9).

Applying Coifman's idea (cf. [6, page 16]), we can construct an approximation to the identity associated with $\mathcal{P}$ on the space of homogeneous type $(\mathbb{R}^{n+1}, \rho, \nu)$, which will be done later in §2, Lemma 2.1. Here and throughout this paper, $V_k(z)$ always denotes the measure $\nu(Q(z, 2^{-k}))$ for $k \in \mathbb{Z}$ and $z \in \mathbb{R}^{n+1}$.

Definition 1.2. Let ρ satisfy condition (1.9). A sequence of operators $\{S_k\}_{k\in\mathbb{Z}}$ is said to be an *approximation to the identity associated with $\mathcal{P}$ on the space of homogeneous type* $(\mathbb{R}^{n+1}, \rho, \nu)$ if there exist positive constants C_1, C_2, C_3 such that, for all $k \in \mathbb{Z}$ and all z, z', w, $w' \in \mathbb{R}^{n+1}$, the kernels $S_k(z, w)$ of S_k satisfy the following conditions:

(i) $S_k(z, w) = 0$ if $\rho(z, w) > C_1 2^{-k}$ (which means that each $S_k(\cdot, w)$ is supported on the section $Q(w, C_1 2^{-k})$ and each $S_k(z, \cdot)$ is supported on the section $Q(z, C_1 2^{-k})$);

(ii) $|S_k(z, w)| \le \dfrac{C_2}{V_k(z) + V_k(w)}$;

(iii) $|S_k(z, w) - S_k(z', w)| \le C_2 \dfrac{(2^k \rho(z, z'))^\varepsilon}{V_k(z) + V_k(w)}$ for $\rho(z, z') \le C_3 2^{-k}$;

(iv) $|S_k(z, w) - S_k(z, w')| \le C_2 \dfrac{(2^k \rho(w, w'))^\varepsilon}{V_k(z) + V_k(w)}$ for $\rho(w, w') \le C_3 2^{-k}$;

(v) $\big|[S_k(z, w) - S_k(z', w)] - [S_k(z, w') - S_k(z', w')]\big| \le C_2 \dfrac{(2^k \rho(z, z'))^\varepsilon (2^k \rho(w, w'))^\varepsilon}{V_k(z) + V_k(w)}$ for $\rho(z, z') \le C_3 2^{-k}$ and $\rho(w, w') \le C_3 2^{-k}$;

(vi) $\displaystyle\int_{\mathbb{R}^{n+1}} S_k(z, w)\, d\nu(z) = 1$ for all $w \in \mathbb{R}^{n+1}$;

(vii) $\displaystyle\int_{\mathbb{R}^{n+1}} S_k(z, w)\, d\nu(w) = 1$ for all $z \in \mathbb{R}^{n+1}$.

Let $D_k = S_k - S_{k-1}$. Applying Coifman's decomposition to the identity, we write

$$I = \Big(\sum_{k=-\infty}^{\infty} D_k\Big)\Big(\sum_{j=-\infty}^{\infty} D_j\Big) = \sum_k \sum_{\{j:|k-j|\le N\}} D_k D_j + \sum_k \sum_{\{j:|k-j|>N\}} D_k D_j =: T_N + R_N.$$

Set $D_k^N := \sum_{|j|\le N} D_{k+j}$. Then both T_N and R_N can be represented as

$$T_N = \sum_k D_k^N D_k = \sum_k D_k D_k^N$$

and

$$R_N = \sum_k \sum_{|j|>N} D_{k+j} D_k = \sum_k \sum_{|j|>N} D_k D_{k+j},$$

respectively. Using Cotlar-Stein almost orthogonal estimates, one obtains a similar Calderón-type reproducing formula

$$f = \sum_{k=-\infty}^{\infty} T_N^{-1} D_k^N D_k(f) = \sum_{k=-\infty}^{\infty} D_k^N D_k T_N^{-1}(f) \tag{1.11}$$

in $L^2(\mathbb{R}^{n+1}, d\nu)$, where N is a fixed large integer and T_N^{-1} is the inverse of T_N. See the argument of (2.4) below. In the next theorem, we will show that this Calderón-type reproducing formula still holds for certain subspace of $L^2(\mathbb{R}^{n+1}, d\nu)$.

Theorem 1.3. *Let $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation to the identity associated with $\mathcal{P}$ on $(\mathbb{R}^{n+1}, \rho, \nu)$, set $D_k = S_k - S_{k-1}$, and ε is the one from* (1.9)*. For $|\alpha| < \frac{\varepsilon}{4}$ and $1 \le p, q \le \infty$, if $f \in L^2(\mathbb{R}^{n+1}, d\nu)$ and satisfies that*

$$\begin{cases} \left(\sum_{k\in\mathbb{Z}} \left(2^{k\alpha} \|D_k(f)\|_{L^p_\nu}\right)^q \right)^{1/q} & \text{for } 1 \le q < \infty \\ \sup_{k\in\mathbb{Z}} 2^{k\alpha} \|D_k(f)\|_{L^p_\nu} & \text{for } q = \infty \end{cases} \tag{1.12}$$

is finite, then (1.11) *holds with respect to the norm defined by* (1.12).

The above theorem leads us to introduce a test function space as follows.

Definition 1.4. Let $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation to the identity associated with $\mathcal{P}$ on $(\mathbb{R}^{n+1}, \rho, \nu)$ and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$. For $|\alpha| < \frac{\varepsilon}{4}$ and $1 \le p, q \le \infty$, define

$$\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}} = \{f \in L^2(\mathbb{R}^{n+1}, d\nu) : \|f\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} < \infty\},$$

where

$$\|f\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} := \begin{cases} \left(\sum_{k\in\mathbb{Z}} \left(2^{k\alpha} \|D_k(f)\|_{L^p_\nu}\right)^q \right)^{1/q} & \text{if } 1 \le q < \infty \\ \sup_{k\in\mathbb{Z}} 2^{k\alpha} \|D_k(f)\|_{L^p_\nu} & \text{if } q = \infty \end{cases}.$$

It is clear that the test function space $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ is a subspace of $L^2(\mathbb{R}^{n+1}, d\nu)$. Applying the above Calderón-type reproducing formula (1.11), one can show that the test function space $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ is independent of the choice of the approximation to the identity (see Theorem 4.1 below). Let $\left(\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}\right)'$ denote the dual of $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$. Note that for each fixed k and x, the function $D_k(x, \cdot)$ belongs to $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ for all $|\alpha| < \frac{\varepsilon}{4}$, $1 \le p, q \le \infty$ (see Lemma 2.5 below), and thus $D_k(f)$ is well defined for all $f \in (\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}})'$. Moreover, applying the second difference smoothness condition of the approximation to the identity associated with $\mathcal{P}$ on $(\mathbb{R}^{n+1}, \rho, \nu)$, we will show that the Calderoń-type reproducing formula (1.11) still holds on dual spaces; that is, the following (1.13) holds.

Theorem 1.5. *Under the same assumptions as Theorem 1.3, for each $f \in \left(\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}\right)'$,*

$$\langle f, g \rangle = \sum_{k\in\mathbb{Z}} \langle T_N^{-1} D_k D_k^N(f), g \rangle = \sum_{k\in\mathbb{Z}} \langle D_k D_k^N T_N^{-1}(f), g \rangle, \qquad \forall\, g \in \dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}. \tag{1.13}$$

We now may define the Besov spaces associated with parabolic sections as follows.

Definition 1.6. For $|\alpha| < \frac{\varepsilon}{4}$ and $1 \le p, q \le \infty$, let p' and q' denote the conjugate index of p and q, respectively. Suppose that $\{S_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity associated with $\mathcal{P}$ on $(\mathbb{R}^{n+1}, \rho, \nu)$ and set $D_k = S_k - S_{k-1}$. The *Besov spaces associated with* $\mathcal{P}$ are defined to be

$$\dot{B}^{\alpha,q}_{p,\mathcal{P}} = \Big\{ f \in \big(\dot{\mathcal{B}}^{-\alpha,q'}_{p',\mathcal{P}}\big)' : \|f\|_{\dot{B}^{\alpha,q}_{p,\mathcal{P}}} < \infty \Big\},$$

where

$$\|f\|_{\dot{B}^{\alpha,q}_{p,\mathcal{P}}} := \begin{cases} \Bigg(\displaystyle\sum_{k\in\mathbb{Z}} \big(2^{k\alpha} \|D_k(f)\|_{L^p_\nu}\big)^q \Bigg)^{1/q} & \text{if } \ 1 \le q < \infty \\ \displaystyle\sup_{k\in\mathbb{Z}} 2^{k\alpha} \|D_k(f)\|_{L^p_\nu} & \text{if } \ q = \infty \end{cases}.$$

It is known that the space of Schwartz functions is dense in the classical Besov space on $\mathbb{R}^n$ (see [18, page 48]). We show that the test function space $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ is dense in $\dot{B}^{\alpha,q}_{p,\mathcal{P}}$ as well.

Theorem 1.7. *Let* $|\alpha| < \frac{\varepsilon}{4}$ *and* $1 \le p, q \le \infty$. *Then*

$$\overline{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} = \dot{B}^{\alpha,q}_{p,\mathcal{P}},$$

where $\overline{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}$ *denotes the closure of* $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ *with respect to* $\|\cdot\|_{\dot{B}^{\alpha,q}_{p,\mathcal{P}}}$.

As usual, we have the duality for $\dot{B}^{\alpha,q}_{p,\mathcal{P}}$ as follows.

Theorem 1.8. *Let* $|\alpha| < \frac{\varepsilon}{4}$.

(a) *For* $1 \le p, q \le \infty$ *and each* $g \in \dot{B}^{-\alpha,q'}_{p',\mathcal{P}}$, *the mapping* $\mathcal{L}_g : f \mapsto \int_{\mathbb{R}^{n+1}} f(x)g(x)d\nu(x)$, *defined initially on* $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$, *extends to a bounded linear functional on* $\dot{B}^{\alpha,q}_{p,\mathcal{P}}$ *and satisfies* $\|\mathcal{L}_g\| \lesssim \|g\|_{\dot{B}^{-\alpha,q'}_{p',\mathcal{P}}}$.

(b) *Conversely, for* $1 \le p, q < \infty$, *every bounded linear functional* $\mathcal{L}$ *on* $\dot{B}^{\alpha,q}_{p,\mathcal{P}}$ *can be realized as* $\mathcal{L} = \mathcal{L}_g$ *with* $g \in \dot{B}^{-\alpha,q'}_{p',\mathcal{P}}$ *and* $\|g\|_{\dot{B}^{-\alpha,q'}_{p',\mathcal{P}}} \lesssim \|\mathcal{L}\|$.

Remark 1.9. *When* $0 < \alpha < \frac{\varepsilon}{4}$ *and* $p = q = \infty$, *it follows from* [13, Theorem 3.1] *that* $\dot{B}^{\alpha,\infty}_{\infty,\mathcal{P}}$ *and* $\mathrm{Lip}^{\alpha}_{\mathcal{P}}$, *the Lipschitz spaces associated with parabolic sections, coincide. It was proved in* [13, Theorem 1.1] *that* $\mathrm{Lip}^{\alpha}_{\mathcal{P}}$ *agree with the Campanato spaces which can be viewed as the duals of Hardy spaces associated with parabolic sections* ([13, Theorem 1.5]). *Therefore, the Besov spaces* $\dot{B}^{\alpha,q}_{p,\mathcal{P}}$ *introduced here generalize Lipschitz spaces* $\mathrm{Lip}^{\alpha}_{\mathcal{P}}$.

Finally, we give an embedding theorem for $\dot{B}^{\alpha,q}_{p,\mathcal{P}}$. To show the embedding theorem, we need a lower bound condition on the measure ν; that is, there exist two positive constants ω and C such that, for any parabolic section $Q \in \mathcal{P}$,

$$Cr^{\omega} \le \nu(Q(z,r)) \qquad \text{for all } r > 0, z \in \mathbb{R}^{n+1}. \tag{1.14}$$

The lower bound conditions on the measure had been intensively studied when the underlying spaces are Riemannian manifolds. To be more precise, let (M,g) be a complete non-compact Riemannian manifold of dimension n having non-negative curvature, and μ denote the canonical Riemannian measure on M. It follows from the celebrated Bishop-Gromov comparison theorem (cf. [3]) that $\mu(B(x,2r)) \le 2^n \mu(B(x,r))$. In this setting, the measure with lower bound condition is related to Sobolev-type inequality, the isoperimetric inequality and Poincaré's inequality. For more details, see [16, Chapter 3.1] (especially Theorems 3.1.1 and 3.1.2). See also [5].

Theorem 1.10. *Suppose that the measure ν satisfies* (1.14). *Let ε be given by* (1.9). *For $-\frac{\varepsilon}{4} < \alpha_1 < \alpha_2 < \frac{\varepsilon}{4}$, $1 \le p_2 < p_1 \le \infty$, $\alpha_2 - \frac{\omega}{p_2} = \alpha_1 - \frac{\omega}{p_1}$, and $1 \le q \le \infty$, the embedding map $\dot{B}^{\alpha_2,q}_{p_2,\mathcal{P}} \hookrightarrow \dot{B}^{\alpha_1,q}_{p_1,\mathcal{P}}$ is continuous.*

The embedding theorem for Besov spaces on spaces of homogeneous type was proved by Han [10] under the assumption $\mu(B(x,r)) \approx r$. It was proved in [9] that if the Sobolev embedding theorem holds in $\Omega \subset \mathbb{R}^n$, in any of possible cases, then Ω satisfies the measure density condition; that is, there exists a constant $c > 0$ such that $|B(x,r) \cap \Omega| \ge cr^n$ for all $x \in \Omega$ and all $0 < r \le 1$. Hence, it is reasonable to add condition (1.14) in our hypothesis.

The organization is as follows. We construct an approximation to the identity associated with $\mathcal{P}$ in the next section. Section 3 is devoted to the proofs of Calderón-type reproducing formulae on test function spaces $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ and its dual. We discuss the dense subspace of Besov spaces $\dot{B}^{\alpha,q}_{p,\mathcal{P}}$ and their duals in section 4. The embedding theorem is proved in the last section. We use $a \wedge b$ and $a \vee b$ to denote $\min\{a,b\}$ and $\max\{a,b\}$, respectively. The notation $f(x) \lesssim g(x)$ is used to indicate that $f(x) \le Cg(x)$ for some $C > 0$. And the notation $f(x) \approx g(x)$ denotes both $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$.

2. Existence of the approximation to the identity

In this section, we construct an approximation of the identity in the sense of Definition 1.2. As mentioned before, the idea comes from Coifman and Weiss. Let $\psi : \mathbb{R} \mapsto [0,1]$ be a smooth function which is 1 on $(-1,1)$ and vanishes on $(-\infty,-2) \cup (2,\infty)$. We define

$$U_k(f)(z) = \int_{\mathbb{R}^{n+1}} \psi(2^k \rho(z,w)) f(w) d\nu(w), \qquad k \in \mathbb{Z}.$$

Let M_k be the operator of multiplication by $M_k(z) := \frac{1}{U_k(1)(z)}$ and W_k be the operator of multiplication by $W_k(z) := \left[U_k\left(\frac{1}{U_k(1)}\right)(z)\right]^{-1}$. Then

(a) $U_k(1)(z) \approx \nu(Q(z,2^{-k})) := V_k(z)$. Indeed,

$$U_k(1)(z) \le \int_{\rho(z,w) \le 2^{1-k}} d\nu(w) \le \nu(Q(z,2^{1-k})) \lesssim \nu(Q(z,2^{-k})).$$

Conversely,

$$U_k(1)(z) \ge \int_{\rho(z,w) < 2^{-k}} d\nu(w) = \nu(Q(z,2^{-k})).$$

(b) $V_k(z) \approx V_k(w)$ whenever $\rho(z,w) \le A^3 2^{5-k}$. Here and in what follows, we always use A to denote the constant given in (1.10).

We prove $V_k(z) \lesssim V_k(w)$ only since the reverse estimate is similar. By (1.8) and (1.9), it is easy to see that $Q(z, 2^{-k}) \subset B_d(z, \theta 2^{1-k}) \subset B_\rho(z, c\theta 2^{1-k})$, where the constant c is given in (1.9). If $\rho(z,w) \le A^3 2^{5-k}$, then for any $\bar{z} \in Q(z, 2^{-k})$,

$$\rho(\bar{z}, w) \le A(\rho(\bar{z}, z) + \rho(z,w)) \le A(c\theta 2^{1-k} + A^3 2^{5-k}) < A(c\theta + A^3)2^{5-k},$$

which implies $Q(z, 2^{-k}) \subset B_\rho(w, A(c\theta + A^3)2^{5-k})$. By (1.8) and (1.9) again, we have

$$Q(z, 2^{-k}) \subset B_d(w, cA(c\theta + A^3)2^{5-k}) \subset Q(w, cA(c\theta + A^3)2^{5-k}).$$

The doubling condition (1.5) of ν with respect to parabolic sections yields

$$V_k(z) \le \nu(Q(w, cA(c\theta + A^3)2^{5-k})) \lesssim \nu(Q(w, 2^{-k})) = V_k(w).$$

(c) $U_k\big(\frac{1}{U_k(1)}\big)(z) \approx 1$ for all $k \in \mathbb{Z}$. Immediately, properties (a) and (b) give

$$\begin{aligned} U_k\left(\frac{1}{U_k(1)}\right)(z) &\approx \int_{\mathbb{R}^{n+1}} \psi(2^k \rho(z,w)) \frac{1}{V_k(w)} d\nu(w) \\ &\approx \frac{1}{V_k(z)} \int_{\mathbb{R}^{n+1}} \psi(2^k \rho(z,w)) d\nu(w) \\ &\approx 1. \end{aligned}$$

Set $S_k = M_k U_k W_k U_k M_k$. Then the kernel of S_k is

$$S_k(z,w) = \int_{\mathbb{R}^{n+1}} M_k(z)\psi(2^k\rho(z,u))W_k(u)\psi(2^k\rho(u,w))M_k(w)d\nu(u), \tag{2.1}$$

where $(z,w) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, and the sequence of operators $\{S_k\}_{k\in\mathbb{Z}}$ is an approximation to the associated with parabolic sections.

Lemma 2.1. *The kernels $S_k(z,w)$ of operators S_k, given by* (2.1)*, satisfy the following properties:*

(i) $S_k(z,w) = S_k(w,z)$;

(ii) $S_k(z,w) = 0$ *if* $\rho(z,w) > A2^{2-k}$ *and* $|S_k(z,w)| \lesssim \dfrac{1}{V_k(z) + V_k(w)}$;

(iii) $|S_k(z,w) - S_k(z',w)| \lesssim \dfrac{(2^k\rho(z,z'))^\varepsilon}{V_k(z) + V_k(w)}$ *for* $\rho(z,z') \le A^3 2^{5-k}$;

(iv) $|S_k(z,w) - S_k(z,w')| \lesssim \dfrac{(2^k\rho(w,w'))^\varepsilon}{V_k(z) + V_k(w)}$ *for* $\rho(w,w') \le A^3 2^{5-k}$;

(v) $\big|[S_k(z,w) - S_k(z',w)] - [S_k(z,w') - S_k(z',w')]\big| \lesssim \dfrac{(2^k\rho(z,z'))^\varepsilon (2^k\rho(w,w'))^\varepsilon}{V_k(z) + V_k(w)}$ *for* $\rho(z,z') \le A^3 2^{5-k}$ *and* $\rho(w,w') \le A^3 2^{5-k}$;

(vi) $\displaystyle\int_{\mathbb{R}^{n+1}} S_k(z,w)d\nu(z) = 1$ *for all* $w \in \mathbb{R}^{n+1}$;

(vii) $\displaystyle\int_{\mathbb{R}^{n+1}} S_k(z,w)d\nu(w) = 1$ *for all* $z \in \mathbb{R}^{n+1}$.

Proof. Property (i) is obvious since $\rho(z,w)=\rho(w,z)$. (ii) If $S_k(z,w)\neq 0$, then $\rho(z,u)\le 2^{1-k}$ and $\rho(u,w)\le 2^{1-k}$, and hence $\rho(z,w)\le A2^{2-k}$. That is, $S_k(z,w)=0$ when $\rho(z,w)>A2^{2-k}$. The definition of M_k and property (c) give

$$\begin{aligned}|S_k(z,w)| &\le \frac{1}{U_k(1)(z)}\frac{1}{U_k(1)(w)}\int_{\rho(z,u)\le 2^{1-k}}\psi(2^k\rho(z,u))W_k(u)\psi(2^k\rho(u,w))d\nu(u)\\ &\lesssim \frac{1}{V_k(z)}\frac{1}{V_k(w)}\nu(Q(z,2^{1-k}))\\ &\lesssim \frac{1}{V_k(w)},\end{aligned}$$

which implies $|S_k(z,w)|\lesssim \frac{1}{V_k(z)+V_k(w)}$ whenever $\rho(z,w)\le A2^{2-k}$.

For (iii), we write

$$\begin{aligned}&S_k(z,w)-S_k(z',w)\\ &=\int_{\mathbb{R}^{n+1}}[M_k(z)\psi(2^k\rho(z,u))-M_k(z')\psi(2^k\rho(z',u))]W_k(u)\psi(2^k\rho(u,w))M_k(w)d\nu(u)\\ &=\int_{\mathbb{R}^{n+1}}[M_k(z)-M_k(z')]\psi(2^k\rho(z,u))W_k(u)\psi(2^k\rho(u,w))M_k(w)d\nu(u)\\ &\quad+\int_{\mathbb{R}^{n+1}}M_k(z')[\psi(2^k\rho(z,u))-\psi(2^k\rho(z',u))]W_k(u)\psi(2^k\rho(u,w))M_k(w)d\nu(u)\\ &:=I_1+I_2.\end{aligned}$$

To estimate I_1, we use property (a) to obtain

$$|M_k(z)-M_k(z')|=\frac{|U_k(1)(z')-U_k(1)(z)|}{U_k(1)(z')U_k(1)(z)}\approx\frac{|U_k(1)(z')-U_k(1)(z)|}{V_k(z')V_k(z)}.$$

By the definition of $U_k(1)(z)$,

$$U_k(1)(z')-U_k(1)(z)=\int_{\mathbb{R}^{n+1}}\psi(2^k\rho(z',w))-\psi(2^k\rho(z,w))d\nu(w).$$

The above integrand $\psi(2^k\rho(z',w))-\psi(2^k\rho(z,w))$ is supported on $B_\rho(z,2^{1-k})\cup B_\rho(z',2^{1-k})$. If $\rho(z,z')\le A^32^{5-k}$, then $B_\rho(z,2^{1-k})\cup B_\rho(z',2^{1-k})\subset B_\rho(z',A^42^{5-k})$ and

$$|U_k(1)(z')-U_k(1)(z)|\le\int_{B_\rho(z',A^42^{5-k})}|\psi(2^k\rho(z',w))-\psi(2^k\rho(z,w))|d\nu(w).$$

Note that, for $w\in B_\rho(z',A^42^{5-k})$, we have $\rho(z,w)\le A(\rho(z,z')+\rho(z',w))<A^52^{6-k}$. Since $|\rho(z,u)-\rho(w,u)|\le c(\rho(z,w))^\varepsilon[\rho(z,u)+\rho(w,u)]^{1-\varepsilon}$,

$$\begin{aligned}|\psi(2^k\rho(z,w))-\psi(2^k\rho(z',w))| &\lesssim 2^k(\rho(z,z'))^\varepsilon[\rho(z,w)+\rho(z',w)]^{1-\varepsilon}\\ &\lesssim 2^k2^{-k(1-\varepsilon)}(\rho(z,z'))^\varepsilon\\ &=(2^k\rho(z,z'))^\varepsilon.\end{aligned}\tag{2.2}$$

For $\rho(z,z')\le A^32^{5-k}$, the above (2.2) and doubling condition of ν give

$$|U_k(1)(z')-U_k(1)(z)|\lesssim(2^k\rho(z,z'))^\varepsilon\nu(B_\rho(z',A^42^{5-k}))\lesssim V_k(z')(2^k\rho(z,z'))^\varepsilon,$$

which yields

$$|M_k(z) - M_k(z')| \lesssim (2^k\rho(z,z'))^\varepsilon \frac{1}{V_k(z)}. \tag{2.3}$$

Hence, the support condition of ψ gives

$$|I_1| \le |M_k(z) - M_k(z')| M_k(w) \int_{B_\rho(z,2^{1-k})\cap B_\rho(w,2^{1-k})} \psi(2^k\rho(z,u)) W_k(u) \psi(2^k\rho(u,w)) d\nu(u).$$

If $\rho(z,w) > A2^{2-k}$, $B_\rho(z,2^{1-k}) \cap B_\rho(w,2^{1-k}) = \varnothing$ implies $I_1 = 0$. If $\rho(z,w) \le A2^{2-k}$, property (b) shows $V_k(z) \approx V_k(w)$, and then

$$|I_1| \lesssim (2^k\rho(z,z'))^\varepsilon \frac{1}{V_k(z)+V_k(w)}.$$

In any case,

$$|I_1| \lesssim (2^k\rho(z,z'))^\varepsilon \frac{1}{V_k(z)+V_k(w)} \qquad \text{provided } \rho(z,z') \le A^3 2^{5-k}.$$

A similar argument to the estimate of I_1 shows that

$$\begin{aligned}
|I_2| &\le M_k(z')M_k(w) \int_{\mathbb{R}^{n+1}} |\psi(2^k\rho(z,u)) - \psi(2^k\rho(z',u))| W_k(u) \psi(2^k\rho(u,w)) d\nu(u) \\
&\lesssim (2^k\rho(z,z'))^\varepsilon \frac{1}{V_k(w)} \\
&\lesssim (2^k\rho(z,z'))^\varepsilon \frac{1}{V_k(z)+V_k(w)} \qquad \text{for } \rho(z,z') \le A^3 2^{5-k}.
\end{aligned}$$

The proof of (iv) is similar to (iii).

To verify (v), we write

$$\begin{aligned}
&[S_k(z,w) - S_k(z',w)] - [S_k(z,w') - S_k(z',w')] \\
&= \int_{\mathbb{R}^{n+1}} [M_k(z)\psi(2^k\rho(z,u)) - M_k(z')\psi(2^k\rho(z',u))] W_k(u) \\
&\qquad \times [\psi(2^k\rho(u,w))M_k(w) - \psi(2^k\rho(u,w'))M_k(w')] d\nu(u) \\
&= \int_{\mathbb{R}^{n+1}} [M_k(z) - M_k(z')] \psi(2^k\rho(z,u)) W_k(u) [\psi(2^k\rho(u,w)) - \psi(2^k\rho(u,w'))] M_k(w) d\nu(u) \\
&\quad + \int_{\mathbb{R}^{n+1}} [M_k(z) - M_k(z')] \psi(2^k\rho(z,u)) W_k(u) \psi(2^k\rho(u,w')) [M_k(w) - M_k(w')] d\nu(u) \\
&\quad + \int_{\mathbb{R}^{n+1}} M_k(z') [\psi(2^k\rho(z,u)) - \psi(2^k\rho(z',u))] W_k(u) \\
&\qquad\quad \times [\psi(2^k\rho(u,w)) - \psi(2^k\rho(u,w'))] M_k(w) d\nu(u) \\
&\quad + \int_{\mathbb{R}^{n+1}} M_k(z') [\psi(2^k\rho(z,u)) - \psi(2^k\rho(z',u))] W_k(u) \psi(2^k\rho(u,w')) [M_k(w) - M_k(w')] d\nu(u) \\
&:= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

To estimate J_1, we use (2.2) and (2.3) for $\rho(z,z') \le A^3 2^{5-k}$ and $\rho(w,w') \le A^3 2^{5-k}$ combined with the support condition of ψ to get

$$|J_1| \lesssim (2^k\rho(z,z'))^\varepsilon (2^k\rho(w,w'))^\varepsilon \frac{1}{V_k(z)+V_k(w)}.$$

Similarly, for $\rho(z,z') \le A^3 2^{5-k}$ and $\rho(w,w') \le A^3 2^{5-k}$,

$$|J_2|+|J_3|+|J_4| \lesssim (2^k\rho(z,z'))^\varepsilon (2^k\rho(w,w'))^\varepsilon \frac{1}{V_k(z)+V_k(w)}.$$

For (vi),

$$\begin{aligned}\int S_k(z,w)d\nu(z) &= \int \Big(\int \psi(2^k\rho(u,z))M_k(z)d\nu(z)\Big) W_k(u)\psi(2^k\rho(u,w))M_k(w)d\nu(u)\\ &= \int \Big[U_k\Big(\frac{1}{U_k(1)}\Big)(u)\Big] W_k(u)\psi(2^k\rho(u,w))M_k(w)d\nu(u)\\ &= M_k(w)\int \psi(2^k\rho(u,w))d\nu(u)\\ &= M_k(w)U_k(1)(w) = 1,\end{aligned}$$

and (vii) is obtained by the same argument. □

Remark 2.2. *According to Lemma 2.1 (ii) and the fact $D_k = S_k - S_{k-1}$, it is easy to check*

$$\int_{\mathbb{R}^{n+1}} |D_k(x,y)|d\nu(x) \lesssim \int_{\rho(x,y)\le A2^{3-k}} \frac{1}{V_k(x)+V_k(y)} d\nu(x) \le C \qquad \text{for each } y,$$

which implies that D_k is bounded on L^1_ν. Similarly,

$$\int_{\mathbb{R}^{n+1}} |D_k(x,y)|d\nu(y) \le C \qquad \text{for each } x,$$

which implies the L^∞_ν-boundedness of D_k. By interpolation, each D_k is bounded on L^p_ν for $1 \le p \le \infty$.

Lemma 2.3. *Let $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation to the identity associated with $\mathcal{P}$ on $(\mathbb{R}^{n+1},\rho,\nu)$ and set $D_k = S_k - S_{k-1}$. Then*

$$|D_jD_k(z,w)| \lesssim 2^{-|j-k|\varepsilon} \frac{1}{V_{j\wedge k}(z)+V_{j\wedge k}(w)} \chi_{\{(z,w):\ \rho(z,w)\le A^2 2^{4-(j\wedge k)}\}}.$$

Proof. By (ii) of Lemma 2.1, it is easy to check that $D_jD_k(z,w) = 0$ whenever $\rho(z,w) > A^2 2^{4-(j\wedge k)}$. For $k \ge j$, we use vanishing condition of D_k and Lemma 2.1 (ii), (iv) to get

$$\begin{aligned}|D_jD_k(z,w)| &\le \int_{\rho(u,w)\le A2^{3-k}} |D_j(z,u)-D_j(z,w)||D_k(u,w)|d\nu(u)\\ &\lesssim \int_{\rho(u,w)\le A2^{3-k}} \Big(2^j\rho(u,w)\Big)^\varepsilon \frac{1}{V_j(w)}\frac{1}{V_k(w)} d\nu(u)\\ &\lesssim 2^{-(k-j)\varepsilon}\frac{1}{V_j(w)}.\end{aligned}$$

Similarly, for $k<j$, the vanishing condition of D_j and Lemma 2.1 (ii), (iii) show

$$\begin{aligned}|D_jD_k(z,w)| &\le \int_{\rho(z,u)\le A2^{3-j}} |D_j(z,u)||D_k(u,w)-D_k(z,w)|d\nu(u)\\ &\lesssim \int_{\rho(z,u)\le A2^{3-j}} \frac{1}{V_j(z)}\Big(2^k\rho(u,z)\Big)^{\varepsilon}\frac{1}{V_k(z)}d\nu(u)\\ &\lesssim 2^{-(j-k)\varepsilon}\frac{1}{V_k(z)}.\end{aligned}$$

Since $V_k(z)\approx V_k(w)$ when $\rho(z,w)\le A^2 2^{4-k}$, the proof is finished. □

By Lemma 2.1 (ii) and Lemma 2.3, we immediately have the following result.

Lemma 2.4. *Let $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation to the identity associated with $\mathcal{P}$ on $(\mathbb{R}^{n+1},\rho,\nu)$ and set $D_k=S_k-S_{k-1}$. For $1\le p\le\infty$, $\|D_jD_k\|_{L^p_\nu\mapsto L^p_\nu}\lesssim 2^{-|j-k|\varepsilon}$. Moreover,*

$$\|D_jD_k^N\|_{L^p_\nu\mapsto L^p_\nu}\lesssim \sum_{|s|\le N} 2^{-|j-k-s|\varepsilon} \qquad \text{and} \qquad \|D_k^ND_j\|_{L^p_\nu\mapsto L^p_\nu}\lesssim \sum_{|s|\le N} 2^{-|j-k-s|\varepsilon}.$$

By plugging $p=2$ into Lemma 2.4, the Cotlar-Stein lemma says

$$\|R_N(f)\|_{L^2_\nu}\lesssim 2^{-N\varepsilon}\|f\|_{L^2_\nu}$$

and then $T_N^{-1}=\sum_{m=0}^{\infty}(R_N)^m$ is bounded on L^2_ν. This yields

$$I=\sum_{k\in\mathbb{Z}}T_N^{-1}D_k^ND_k=\sum_{k\in\mathbb{Z}}D_k^ND_kT_N^{-1} \qquad \text{in } L^2_\nu, \tag{2.4}$$

which is (1.11).

To see that $D_k(f)$ is well-defined for $f\in(\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}})'$, we need the following lemma.

Lemma 2.5. *Let $\{S_j\}_{j\in\mathbb{Z}}$ be an approximation to the identity associated with $\mathcal{P}$ on $(\mathbb{R}^{n+1},\rho,\nu)$ and $D_j=S_j-S_{j-1}$. For $|\alpha|<\frac{\varepsilon}{4}$ and $1\le p,q\le\infty$, both $D_j(\cdot,w)$ and $D_j(z,\cdot)$ belong to $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ for all $z,w\in\mathbb{R}^{n+1}$ and $j\in\mathbb{Z}$.*

Proof. Since $D_j(\cdot,w)=D_j(w,\cdot)$ for any fixed $w\in\mathbb{R}^{n+1}$, it suffices to verify the lemma for $D_j(\cdot,w)$. Note that

$$D_\ell(D_j(\cdot,w))(z)=\int_{\mathbb{R}^{n+1}} D_\ell(z,u)D_j(u,w)d\nu(u)=D_\ell D_j(z,w).$$

By Lemma 2.3,

$$\|D_\ell(D_j(\cdot,w))\|_{L^\infty_\nu}\lesssim 2^{-|j-\ell|\varepsilon}\frac{1}{V_j(w)}$$

and

$$\|D_\ell(D_j(\cdot,w))\|_{L^1_\nu}\lesssim 2^{-|j-\ell|\varepsilon}.$$

For $1<p<\infty$, the interpolation theorem implies

$$\|D_\ell(D_j(\cdot,w))\|_{L^p_\nu}\le \|D_\ell(D_j(\cdot,w))\|_{L^\infty_\nu}^{1-\frac{1}{p}}\|D_\ell(D_j(\cdot,w))\|_{L^1_\nu}^{\frac{1}{p}}\lesssim 2^{-|j-\ell|\varepsilon}V_j(w)^{\frac{1}{p}-1}.$$

Combining above estimates, we obtain

$$\begin{aligned}\|D_j(\cdot,w)\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} &= \left\{\sum_{\ell\in\mathbb{Z}}\left(2^{\ell\alpha}\|D_\ell(D_j(\cdot,w))\|_{L^p_\nu}\right)^q\right\}^{\frac{1}{q}}\\ &\lesssim (V_j(w))^{\frac{1}{p}-1}\left\{\sum_{\ell\in\mathbb{Z}}2^{\ell\alpha q}2^{-|j-\ell|\varepsilon q}\right\}^{\frac{1}{q}}\\ &= 2^{j\alpha}(V_j(w))^{\frac{1}{p}-1}\left\{\sum_{\ell\le j}2^{-(j-\ell)(\alpha+\varepsilon)q}+\sum_{\ell>j}2^{(\ell-j)(\alpha-\varepsilon)q}\right\}^{\frac{1}{q}}\\ &\lesssim 2^{j\alpha}(V_j(w))^{\frac{1}{p}-1}\end{aligned}$$

and the proof follows. □

Remark 2.6. *Using the same argument in the proof of Lemma 2.5, we can show that if* $f\in C^1(\mathbb{R}^{n+1})$ *with compact support and*

$$\int_{\mathbb{R}^{n+1}} f(z)d\nu(z)=0,$$

then $f\in\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ *for* $|\alpha|<\frac{\varepsilon}{4}$ *and* $1\le p,q\le\infty$.

3. Calderón-type reproducing formulae for $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ and their duals

In this section, we are going to show Theorems 1.3 and 1.5, which are the Calderón-type reproducing formula for $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ and their duals, respectively.

Proof of Theorem 1.3. We prove the first equality in (1.11) in $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ only because the proof for the second one is similar. Choose a large number $N\in\mathbb{N}$ at least satisfying $\frac{2}{1-2^{-\varepsilon/4}}2^{-N\varepsilon/4}<1$. We claim that there exists $C_0>0$ such that

$$\|R_N(f)\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}\le C_0N^{\frac{3}{2}}2^{-N(\frac{\varepsilon}{4}-|\alpha|)}\|f\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}. \tag{3.1}$$

Since $f=\sum_{k'\in\mathbb{Z}}T_N^{-1}D^N_{k'}D_{k'}(f)$ in L^2_ν (N will be chosen later), there is a subsequence (written in the same indices for simplicity) convergence almost everywhere and hence

$$\begin{aligned}D_kR_N(f)(z) &= D_kR_N\left(\sum_{k'\in\mathbb{Z}}T_N^{-1}D^N_{k'}D_{k'}(f)\right)(z)\\ &= D_kR_N\sum_{k'\in\mathbb{Z}}\sum_{m=0}^{\infty}(R_N)^mD^N_{k'}D_{k'}(f)(z)\\ &= \sum_{k'\in\mathbb{Z}}\sum_{m=0}^{\infty}D_k(R_N)^{m+1}D^N_{k'}D_{k'}(f)(z).\end{aligned}\tag{3.2}$$

Plugging $R_N=\sum_{k\in\mathbb{Z}}\sum_{|\ell|>N}D_{k+\ell}D_k$, we rewrite

$D_k(R_N)^{m+1}D^N_{k'}$

$$= D_k\Big(\sum_{k_0\in\mathbb{Z}}\sum_{|\ell_0|>N} D_{k_0+\ell_0}D_{k_0}\Big)\cdots\Big(\sum_{k_m\in\mathbb{Z}}\sum_{|\ell_m|>N} D_{k_m+\ell_m}D_{k_m}\Big)D^N_{k'}$$
$$= \sum_{k_0\in\mathbb{Z}}\sum_{|\ell_0|>N}\sum_{k_1\in\mathbb{Z}}\sum_{|\ell_1|>N}\cdots\sum_{k_m\in\mathbb{Z}}\sum_{|\ell_m|>N} D_kD_{k_0+\ell_0}D_{k_0}D_{k_1+\ell_1}D_{k_1}\cdots D_{k_m+\ell_m}D_{k_m}D^N_{k'}.$$

Lemma 2.4 gives

$$\|D_kD_{k_0+\ell_0}D_{k_0}D_{k_1+\ell_1}D_{k_1}\cdots D_{k_{m-1}}D_{k_m+\ell_m}D_{k_m}D^N_{k'}\|_{L^p_\nu\mapsto L^p_\nu}$$
$$\lesssim 2^{-|k-k_0-\ell_0|\varepsilon}2^{-|k_0-k_1-\ell_1|\varepsilon}\cdots 2^{-|k_{m-1}-k_m-\ell_m|\varepsilon}\Big(\sum_{|s|\le N}2^{-|k_m-k'-s|\varepsilon}\Big).$$

On the other hand, Remark 2.2 shows

$$\|D_kD_{k_0+\ell_0}D_{k_0}D_{k_1+\ell_1}D_{k_1}\cdots D_{k_{m-1}}D_{k_m+\ell_m}D_{k_m}D^N_{k'}\|_{L^p_\nu\mapsto L^p_\nu}$$
$$\lesssim N\|D_{k_0+\ell_0}D_{k_0}D_{k_1+\ell_1}D_{k_1}\cdots D_{k_{m-1}}D_{k_m+\ell_m}D_{k_m}\|_{L^p_\nu\mapsto L^p_\nu}$$
$$\lesssim N2^{-|\ell_0|\varepsilon}2^{-|\ell_1|\varepsilon}\cdots 2^{-|\ell_{m-1}|\varepsilon}2^{-|\ell_m|\varepsilon}.$$

Taking the geometric average of these two estimates, we get

$$\|D_kD_{k_0+\ell_0}D_{k_0}D_{k_1+\ell_1}D_{k_1}\cdots D_{k_m+\ell_m}D_{k_m}D^N_{k'}\|_{L^p_\nu\mapsto L^p_\nu}$$
$$\lesssim N^{\frac12}2^{-|k-k_0-\ell_0|\frac{\varepsilon}{2}}2^{-|\ell_0|\frac{\varepsilon}{2}}\cdots 2^{-|k_{m-1}-k_m-\ell_m|\frac{\varepsilon}{2}}2^{-|\ell_m|\frac{\varepsilon}{2}}\Big(\sum_{|s|\le N}2^{-|k_m-k'-s|\varepsilon}\Big)^{\frac12}.$$

Hence,

$$\begin{aligned}\|D_k(R_N)^{m+1}D^N_{k'}\|_{L^p_\nu\mapsto L^p_\nu} &\lesssim N^{\frac12}\sum_{k_0\in\mathbb{Z}}\sum_{|\ell_0|>N}\cdots\sum_{k_m\in\mathbb{Z}}\sum_{|\ell_m|>N}\sum_{|s|\le N}2^{-|k-k_0-\ell_0|\frac{\varepsilon}{2}}\\ &\qquad\times 2^{-|\ell_0|\frac{\varepsilon}{2}}\cdots 2^{-|k_{m-1}-k_m-\ell_m|\frac{\varepsilon}{2}}2^{-|\ell_m|\frac{\varepsilon}{2}}2^{-|k_m-k'-s|\frac{\varepsilon}{2}}\\ &\lesssim N^{\frac12}\sum_{|s|\le N}2^{-|k-k'-s|\frac{\varepsilon}{4}}\Big(\frac{2}{1-2^{-\frac{\varepsilon}{4}}}2^{-\frac{N\varepsilon}{4}}\Big)^{m+1}.\end{aligned}\tag{3.3}$$

Since $\frac{2}{1-2^{-\frac{\varepsilon}{4}}}2^{-\frac{N\varepsilon}{4}}<1$, both (3.2) and (3.3) give

$$\begin{aligned}\|D_kR_N(f)\|_{L^p_\nu} &\le \sum_{k'\in\mathbb{Z}}\sum_{m=0}^{\infty}\|D_k(R_N)^{m+1}D^N_{k'}\|_{L^p_\nu\mapsto L^p_\nu}\|D_{k'}(f)\|_{L^p_\nu}\\ &\lesssim N^{\frac12}\sum_{k'\in\mathbb{Z}}\sum_{m=0}^{\infty}\sum_{|s|\le N}2^{-|k-k'-s|\frac{\varepsilon}{4}}\Big(\frac{2}{1-2^{-\frac{\varepsilon}{4}}}2^{-\frac{N\varepsilon}{4}}\Big)^{m+1}\|D_{k'}(f)\|_{L^p_\nu}\\ &\lesssim N^{\frac12}2^{-\frac{N\varepsilon}{4}}\sum_{k'\in\mathbb{Z}}\sum_{|s|\le N}2^{-|k-k'-s|\frac{\varepsilon}{4}}\|D_{k'}(f)\|_{L^p_\nu}.\end{aligned}\tag{3.4}$$

Therefore, for $1\le q<\infty$,

$$\|R_N(f)\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}\lesssim N^{\frac12}2^{-\frac{N\varepsilon}{4}}\Big\{\sum_{k\in\mathbb{Z}}\Big(2^{k\alpha}\sum_{k'\in\mathbb{Z}}\sum_{|s|\le N}2^{-|k-k'-s|\frac{\varepsilon}{4}}\|D_{k'}(f)\|_{L^p_\nu}\Big)^q\Big\}^{\frac1q}$$

$$= N^{\frac{1}{2}} 2^{-\frac{N\varepsilon}{4}} \Big\{ \sum_{k\in\mathbb{Z}} \Big(\sum_{k'\in\mathbb{Z}} \sum_{|s|\le N} 2^{-|k-k'-s|\frac{\varepsilon}{4}+(k-k'-s)\alpha} 2^{s\alpha} 2^{k'\alpha} \|D_{k'}(f)\|_{L^p_\nu} \Big)^q \Big\}^{\frac{1}{q}}.$$

Hölder's inequality gives

$$\|R_N(f)\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} \lesssim N^{\frac{1}{2}} 2^{-\frac{N\varepsilon}{4}} \Big\{ \sum_{k\in\mathbb{Z}} \Big(\sum_{k'\in\mathbb{Z}} \sum_{|s|\le N} 2^{-|k-k'-s|\frac{\varepsilon}{4}+(k-k'-s)\alpha} 2^{s\alpha} \Big)^{\frac{q}{q'}}$$
$$\times \Big(\sum_{k'\in\mathbb{Z}} \sum_{|s|\le N} 2^{-|k-k'-s|\frac{\varepsilon}{4}+(k-k'-s)\alpha} 2^{s\alpha} (2^{k'\alpha} \|D_{k'}(f)\|_{L^p_\nu})^q \Big) \Big\}^{\frac{1}{q}}.$$

(In case $q = 1$, the part $\big(\sum_{k'\in\mathbb{Z}} \sum_{|s|\le N} 2^{-|k-k'-s|\frac{\varepsilon}{4}+(k-k'-s)\alpha} 2^{s\alpha}\big)^{q/q'}$ is understood to equal 1 and the same remark applies in similar places later on.) Since $|\alpha| < \frac{\varepsilon}{4}$,

$$\sum_{k'\in\mathbb{Z}} 2^{-|k-k'-s|\frac{\varepsilon}{4}+(k-k'-s)\alpha} \le C$$

and then

$$\begin{aligned} \|R_N(f)\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} &\le CN^{\frac{1}{2}} 2^{-\frac{N\varepsilon}{4}} \Big(\sum_{|s|\le N} 2^{s\alpha} \Big) \Big\{ \sum_{k'\in\mathbb{Z}} \big(2^{k'\alpha} \|D_{k'}(f)\|_{L^p_\nu} \big)^q \Big\}^{\frac{1}{q}} \\ &\le C_1 N^{\frac{3}{2}} 2^{-N(\frac{\varepsilon}{4}-|\alpha|)} \|f\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}. \end{aligned} \tag{3.5}$$

While $q = \infty$, inequality (3.4) implies

$$\begin{aligned} 2^{k\alpha} \|D_k R_N(f)\|_{L^p_\nu} &\le CN^{\frac{1}{2}} 2^{-\frac{N\varepsilon}{4}} \sum_{k'\in\mathbb{Z}} \sum_{|s|\le N} 2^{-|k-k'-s|\frac{\varepsilon}{4}} 2^{(k-k'-s)\alpha} 2^{s\alpha} 2^{k'\alpha} \|D_{k'}(f)\|_{L^p_\nu} \\ &\le CN^{\frac{1}{2}} 2^{-\frac{N\varepsilon}{4}} \Big(\sum_{|s|\le N} 2^{s\alpha} \Big) \sup_{k'\in\mathbb{Z}} 2^{k'\alpha} \|D_{k'}(f)\|_{L^p_\nu} \\ &\le C_2 N^{\frac{3}{2}} 2^{-N(\frac{\varepsilon}{4}-|\alpha|)} \|f\|_{\dot{\mathcal{B}}^{\alpha,\infty}_{p,\mathcal{P}}} \qquad \text{for all } k \in \mathbb{Z}. \end{aligned} \tag{3.6}$$

Hence claim (3.1) is proved by setting $C_0 = \max\{C_1, C_2\}$, where the constants C_1 and C_2 are given in (3.5) and (3.6), respectively. We now choose a bigger N such that

$$\max\Big\{ \frac{2}{1-2^{-\frac{\varepsilon}{4}}} 2^{-\frac{N\varepsilon}{4}}, C_0 N^{\frac{3}{2}} 2^{-N(\frac{\varepsilon}{4}-|\alpha|)} \Big\} < 1.$$

Notice that $T_N^{-1} = (I - R_N)^{-1} = \sum_{m=0}^{\infty} (R_N)^m$, so (3.1) implies

$$\|T_N^{-1}(f)\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} \le \frac{1}{1 - C_0 N^{\frac{3}{2}} 2^{-N(\frac{\varepsilon}{4}-|\alpha|)}} \|f\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} := \lambda_N \|f\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}. \tag{3.7}$$

Then $\sum_{k\in\mathbb{Z}} T_N^{-1} D_k^N D_k(f)$ belongs to $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ for $f \in \dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$. In order to prove that $\sum_{k\in\mathbb{Z}} T_N^{-1} D_k^N D_k(f)$ converges to f in $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$, we observe

$$f(x) - \sum_{|k|\le M} T_N^{-1} D_k^N D_k(f)(x) = \sum_{|k|>M} T_N^{-1} D_k^N D_k(f)(x) \qquad \text{for } f \in L^2_\nu.$$

Thus, we only need to make sure that

$$\lim_{M\to\infty} \Big\| \sum_{|k|>M} T_N^{-1} D_k^N D_k(f) \Big\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} = 0. \tag{3.8}$$

For $1 \le p \le \infty$ and $1 \le q < \infty$, by (3.7),

$$\Big\| \sum_{|k|>M} T_N^{-1} D_k^N D_k(f) \Big\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} \lesssim \lambda_N \Big\{ \sum_{\ell\in\mathbb{Z}} 2^{\ell\alpha q} \Big(\sum_{|k|>M} \|D_\ell D_k^N D_k(f)\|_{L^p_\nu} \Big)^q \Big\}^{\frac{1}{q}}$$
$$\le \lambda_N \Big\{ \sum_{\ell\in\mathbb{Z}} 2^{\ell\alpha q} \Big(\sum_{|k|>M} \|D_\ell D_k^N\|_{L^p_\nu \mapsto L^p_\nu} \|D_k(f)\|_{L^p_\nu} \Big)^q \Big\}^{\frac{1}{q}}.$$

Lemma 2.4 and Hölder's inequality imply

$$\Big\| \sum_{|k|>M} T_N^{-1} D_k^N D_k(f) \Big\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} \lesssim \lambda_N \Big\{ \sum_{\ell\in\mathbb{Z}} \Big(\sum_{|k|>M} \sum_{|s|\le N} 2^{-|\ell-k-s|\varepsilon} 2^{(\ell-k-s)\alpha} 2^{s\alpha} 2^{k\alpha} \|D_k(f)\|_{L^p_\nu} \Big)^q \Big\}^{\frac{1}{q}}$$
$$\lesssim \lambda_N \Big\{ \sum_{\ell\in\mathbb{Z}} \Big(\sum_{|k|>M} \sum_{|s|\le N} 2^{-|\ell-k-s|\varepsilon} 2^{(\ell-k-s)\alpha} 2^{s\alpha} \Big)^{\frac{q}{q'}}$$
$$\times \Big(\sum_{|k|>M} \sum_{|s|\le N} 2^{-|\ell-k-s|\varepsilon} 2^{(\ell-k-s)\alpha} 2^{s\alpha} 2^{k\alpha q} \|D_k(f)\|^q_{L^p_\nu} \Big) \Big\}^{\frac{1}{q}}.$$

Therefore,

$$\Big\| \sum_{|k|>M} T_N^{-1} D_k^N D_k(f) \Big\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} \lesssim \lambda_N \Big(\sum_{|s|\le N} 2^{s\alpha} \Big) \Big\{ \sum_{|k|>M} 2^{k\alpha q} \|D_k(f)\|^q_{L^p_\nu} \Big\}^{\frac{1}{q}}$$
$$\lesssim N 2^{N|\alpha|} \lambda_N \Big\{ \sum_{|k|>M} 2^{k\alpha q} \|D_k(f)\|^q_{L^p_\nu} \Big\}^{\frac{1}{q}}.$$

The assumption of $f \in \dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ shows that the right hand side of the above inequality goes to 0 as $M \to \infty$. Thus, the first equality in (1.11) holds for $1 \le q < \infty$. If $q = \infty$, by the monotonicity of ℓ^q and the fact $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}} \subset \dot{\mathcal{B}}^{\alpha,\infty}_{p,\mathcal{P}}$, we also have

$$\lim_{M\to\infty} \Big\| \sum_{|k|>M} T_N^{-1} D_k^N D_k(f) \Big\|_{\dot{\mathcal{B}}^{\alpha,\infty}_{p,\mathcal{P}}} = 0.$$

Hence, the proof is finished. □

We now prove Theorem 1.5.

Proof of Theorem 1.5. For $g \in \dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ and $f \in \big(\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}\big)'$, Theorem 1.3 says

$$\langle f, g \rangle = \Big\langle f, \sum_{k\in\mathbb{Z}} T_N^{-1} D_k^N D_k(g) \Big\rangle = \sum_{k\in\mathbb{Z}} \big\langle f, T_N^{-1} D_k^N D_k(g) \big\rangle, \tag{3.9}$$

where $T_N^{-1} = (I - R_N)^{-1} = \sum_{m=0}^{\infty} (R_N)^m$, $R_N = \sum_{|k-j|>N} D_k D_j$, and $D_k^N = \sum_{|j|\le N} D_{j+k}$. Since these T_N^{-1}, R_N and D_k^N are combinations of D_k, it suffices to claim

$$\big\langle f, D_k(g) \big\rangle = \big\langle D_k(f), g \big\rangle \qquad \text{for } g \in \dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}},\ f \in \big(\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}\big)'. \tag{3.10}$$

Assuming the claim for the moment, we have

$$\big\langle f, D_{k'+\ell} D_{k'} (R_N)^{m-1} D_k^N D_k(g) \big\rangle = \big\langle D_{k'+\ell}(f), D_{k'} (R_N)^{m-1} D_k^N D_k(g) \big\rangle$$

$$= \big\langle D_{k'} D_{k'+\ell}(f), (R_N)^{m-1} D_k^N D_k(g) \big\rangle.$$

Since R_N can be expressed to be $R_N = \sum_{k'\in\mathbb{Z}} \sum_{|\ell|>N} D_{k'+\ell} D_{k'} = \sum_{k'\in\mathbb{Z}} \sum_{|\ell|>N} D_{k'} D_{k'+\ell}$, we take the summation $\sum_{k'\in\mathbb{Z}} \sum_{|\ell|>N}$ on both sides to obtain

$$\big\langle f, R_N(R_N)^{m-1} D_k^N D_k(g) \big\rangle = \big\langle R_N(f), (R_N)^{m-1} D_k^N D_k(g) \big\rangle.$$

Repeating the same process m times, we obtain

$$\big\langle f, T_N^{-1} D_k^N D_k(g) \big\rangle = \big\langle T_N^{-1}(f), D_k^N D_k(g) \big\rangle$$

and then

$$\big\langle f, T_N^{-1} D_k^N D_k(g) \big\rangle = \big\langle D_k D_k^N T_N^{-1}(f), g \big\rangle,$$

which and (3.9) give us

$$\langle f, g \rangle = \sum_{k\in\mathbb{Z}} \big\langle D_k D_k^N T_N^{-1}(f), g \big\rangle.$$

The first equality of (1.13) can be obtained similarly.

We now return to the proof of claim (3.10), which contains three steps:

Step 1. Show that each D_k is bounded on $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ for all $|\alpha| < \frac{\varepsilon}{4}$ and $1 \le p, q \le \infty$.
Step 2. Show that $\langle f, D_k(g) \rangle = \langle D_k(f), g \rangle$ for all $f \in (\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}})'$ and $g \in \dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}} \cap L^p_\nu$.
Step 3. Show that $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}} \subset \overline{L^p_\nu \cap \dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}$, where $\overline{L^p_\nu \cap \dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}$ denotes the closure of $L^p_\nu \cap \dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ with respect to $\|\cdot\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}$.

To prove **step 1**, we use Theorem 1.3 to write

$$\begin{aligned}
\|D_k(f)\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} &= \Big\{ \sum_{\ell\in\mathbb{Z}} 2^{\ell\alpha q} \Big\| D_\ell D_k \Big(\sum_{k'\in\mathbb{Z}} D_{k'}^N D_{k'} T_N^{-1}(f) \Big) \Big\|_{L^p_\nu}^q \Big\}^{\frac{1}{q}} \\
&\le \Big\{ \sum_{\ell\in\mathbb{Z}} 2^{\ell\alpha q} \Big(\sum_{k'\in\mathbb{Z}} \|D_\ell D_k D_{k'}^N\|_{L^p_\nu \mapsto L^p_\nu} \|D_{k'} T_N^{-1}(f)\|_{L^p_\nu} \Big)^q \Big\}^{\frac{1}{q}}.
\end{aligned}$$

Lemma 2.4 and Remark 2.2 give

$$\|D_\ell D_k D_{k'}^N\|_{L^p_\nu \mapsto L^p_\nu} \lesssim N 2^{-|\ell-k|\varepsilon}$$

and

$$\|D_\ell D_k D_{k'}^N\|_{L^p_\nu \mapsto L^p_\nu} \lesssim \sum_{|s|\le N} 2^{-|k-k'-s|\varepsilon}.$$

Taking the geometric average of these two estimates yields

$$\begin{aligned}
\|D_\ell D_k D_{k'}^N\|_{L^p_\nu \mapsto L^p_\nu} &\lesssim N^{\frac{1}{2}} 2^{-|\ell-k|\frac{\varepsilon}{2}} \Big(\sum_{|s|\le N} 2^{-|k-k'-s|\varepsilon} \Big)^{\frac{1}{2}} \\
&\le N^{\frac{1}{2}} \sum_{|s|\le N} 2^{-|\ell-k|\frac{\varepsilon}{2}} 2^{-|k-k'-s|\frac{\varepsilon}{2}} \\
&\le N^{\frac{1}{2}} \sum_{|s|\le N} 2^{-|\ell-k'-s|\frac{\varepsilon}{2}}.
\end{aligned} \tag{3.11}$$

For $1 \le q < \infty$, Hölder's inequality and (3.7) show that

$$\begin{aligned}
\|D_k(f)\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} &\lesssim N^{\frac{1}{2}} \Big\{ \sum_{\ell\in\mathbb{Z}} 2^{\ell\alpha q} \Big(\sum_{k'\in\mathbb{Z}} \sum_{|s|\le N} 2^{-|\ell-k'-s|\frac{\varepsilon}{2}} \|D_{k'}T_N^{-1}(f)\|_{L^p_\nu} \Big)^q \Big\}^{\frac{1}{q}} \\
&\lesssim N^{\frac{1}{2}} \Big(\sum_{|s|\le N} 2^{s\alpha} \Big) \Big\{ \sum_{k'\in\mathbb{Z}} 2^{k'\alpha q} \|D_{k'}T_N^{-1}(f)\|^q_{L^p_\nu} \Big\}^{\frac{1}{q}} \\
&\lesssim N^{\frac{3}{2}} 2^{N|\alpha|} \|T_N^{-1}(f)\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} \\
&\lesssim N^{\frac{3}{2}} 2^{N|\alpha|} \lambda_N \|f\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}.
\end{aligned}$$

If $q = \infty$, using Theorem 1.3, (3.11) and (3.7), we get

$$\begin{aligned}
2^{\ell\alpha} \|D_\ell D_k(f)\|_{L^p_\nu} &\lesssim N^{\frac{1}{2}} \sum_{|s|\le N} 2^{s\alpha} \sum_{k'\in\mathbb{Z}} 2^{(\ell-k'-s)\alpha - |\ell-k'-s|\frac{\varepsilon}{2}} 2^{k'\alpha} \|D_{k'}T_N^{-1}(f)\|_{L^p_\nu} \\
&\lesssim N^{\frac{3}{2}} 2^{N|\alpha|} \sup_{k'\in\mathbb{Z}} 2^{k'\alpha} \|D_{k'}T_N^{-1}(f)\|_{L^p_\nu} \\
&= N^{\frac{3}{2}} 2^{N|\alpha|} \|T_N^{-1}(f)\|_{\dot{\mathcal{B}}^{\alpha,\infty}_{p,\mathcal{P}}} \\
&\lesssim N^{\frac{3}{2}} 2^{N|\alpha|} \lambda_N \|f\|_{\dot{\mathcal{B}}^{\alpha,\infty}_{p,\mathcal{P}}},
\end{aligned}$$

and hence

$$\|D_k(f)\|_{\dot{\mathcal{B}}^{\alpha,\infty}_{p,\mathcal{P}}} \lesssim N^{\frac{3}{2}} 2^{N|\alpha|} \lambda_N \|f\|_{\dot{\mathcal{B}}^{\alpha,\infty}_{p,\mathcal{P}}}.$$

To show **step 2**, for $g \in \dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}} \cap L^p_\nu$, we define

$$g_{k,M}(x) = \int_{Q(0,M)} D_k(x,y) g(y) d\nu(y), \qquad M > 0,$$

where $Q(0,M)$ denotes the section $\{y \in \mathbb{R}^{n+1} : \rho(0,y) < M\}$. By step 1,

$$\|D_k(g) - g_{k,M}\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} = \|D_k(g\chi_{\mathbb{R}^{n+1}\setminus Q(0,M)})\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} \lesssim N^{\frac{3}{2}} 2^{N|\alpha|} \lambda_N \|g\chi_{\mathbb{R}^{n+1}\setminus Q(0,M)}\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}.$$

We claim that

$$\lim_{M\to\infty} \|g\chi_{\mathbb{R}^{n+1}\setminus Q(0,M)}\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} = 0.$$

Indeed, by Remark 2.2 and Lebesgue dominated convergence theorem,

$$\|D_k(g\chi_{\mathbb{R}^{n+1}\setminus Q(0,M)})\|_{L^p_\nu} \lesssim \|g\chi_{\mathbb{R}^{n+1}\setminus Q(0,M)}\|_{L^p_\nu} \to 0 \qquad \text{as} \;\; M \to \infty.$$

For $\rho(x,y) \le A2^{3-k}$ and $\rho(0,y) \ge M$, the triangle inequality (1.10) implies

$$\rho(0,x) \ge \frac{1}{A}\rho(0,y) - \rho(x,y) \ge \frac{M}{A} - A2^{3-k},$$

which yields

$$\begin{aligned}
|D_k(g\chi_{\mathbb{R}^{n+1}\setminus Q(0,M)})(x)| &= \Big| \int_{\mathbb{R}^{n+1}\setminus Q(0,M)} D_k(x,y) g(y) d\nu(y) \Big| \\
&\le |D_k(g)(x)| \chi_{\mathbb{R}^{n+1}\setminus Q(0,\frac{M}{A}-A2^{3-k})}(x) \\
&\le |D_k(g)(x)|.
\end{aligned}$$

Thus, $\|D_k(g\chi_{\mathbb{R}^{n+1}\backslash Q(0,M)})\|_{L^p_\nu} \le \|D_k(g)\|_{L^p_\nu}$ and the series $\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|D_k(g\chi_{\mathbb{R}^{n+1}\backslash Q(0,M)})\|^q_{L^p_\nu}$ converges. Hence, given $\varepsilon > 0$, there exists a large number K such that

$$\sum_{|k|>K} 2^{k\alpha q}\|D_k(g\chi_{\mathbb{R}^{n+1}\backslash Q(0,M)})\|^q_{L^p_\nu} < \varepsilon.$$

On the other hand,

$$\begin{aligned}\sum_{|k|\le K} 2^{k\alpha q}\|D_k(g\chi_{\mathbb{R}^{n+1}\backslash Q(0,M)})\|^q_{L^p_\nu} &\lesssim \sum_{|k|\le K} 2^{k\alpha q}\|g\chi_{\mathbb{R}^{n+1}\backslash Q(0,M)}\|^q_{L^p_\nu}\\ &= \|g\chi_{\mathbb{R}^{n+1}\backslash Q(0,M)}\|^q_{L^p_\nu}\frac{2^{-K\alpha q}\big(1-2^{\alpha q(2K+1)}\big)}{1-2^{\alpha q}}\\ &\to 0 \qquad \text{as } M\to\infty.\end{aligned}$$

Then

$$\lim_{M\to\infty}\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|D_k(g\chi_{\mathbb{R}^{n+1}\backslash Q(0,M)})\|^q_{L^p_\nu} = 0$$

and the claim is proved. Therefore,

$$\langle f, D_k(g)\rangle = \lim_{M\to\infty}\langle f, g_{k,M}\rangle. \tag{3.12}$$

Since $\{\mathrm{int}\, Q(z, 2^{-(k+J)})\}_{z\in Q(0,M)}$ is an open covering of $Q(0,M)$, there exist finite many sections $\{Q(z_j, 2^{-(k+J)})\}_{j=1}^{N_J}$, $z_j \in Q(0,M)$, such that $Q(0,M) \subset \bigcup_{j=1}^{N_J} Q(z_j, 2^{-(k+J)})$. Let

$$\begin{aligned}&Q_1 = Q(0,M)\bigcap Q(z_1, 2^{-(k+J)});\\ &Q_2 = Q(0,M)\bigcap Q(z_2, 2^{-(k+J)})\backslash Q_1;\\ &Q_3 = Q(0,M)\bigcap Q(z_3, 2^{-(k+J)})\backslash(Q_1\cup Q_2);\\ &\qquad\vdots\\ &Q_{N_J} = Q(0,M)\bigcap Q(z_{N_J}, 2^{-(k+J)})\backslash \textstyle\bigcup_{j=1}^{N_J-1} Q_j.\end{aligned}$$

Then $\{Q_j\}_{j=1}^{N_J}$ are disjoint and $\bigcup_{j=1}^{N_J} Q_j = Q(0,M)$. Now we write

$$\begin{aligned}g_{k,M}(x) &= \sum_{j=1}^{N_J}\int_{Q_j} D_k(x,y)g(y)d\nu(y)\\ &= \sum_{j=1}^{N_J}\int_{Q_j}(D_k(x,y)-D_k(x,y_j))g(y)d\nu(y) + \sum_{j=1}^{N_J} D_k(x,y_j)\int_{Q_j} g(y)d\nu(y)\\ &:= g^1_{k,M,J}(x) + g^2_{k,M,J}(x),\end{aligned}$$

where y_j is any point in Q_j. To consider $\|g^1_{k,M,J}\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}$, the second difference smoothness condition (v) in Lemma 2.1 will be used. For simplicity, we denote by

$$H_{k,j}(x,y) = (D_k(x,y) - D_k(x,y_j))\chi_{Q_j}(y).$$

Then $H_{k,j}(x,y)$ satisfies the following conditions

(a) $\mathrm{supp}\, H_{k,j}(\cdot,y) \subset Q(y, 16A^2 2^{-k})$ and $\mathrm{supp}\, H_{k,j}(x,\cdot) \subset Q(x, 8A2^{-k})$;

(b) $\displaystyle\int_{\mathbb{R}^{n+1}} H_{k,j}(x,y)d\nu(x) = \chi_{Q_j}(y)\int\big(D_k(x,y)-D_k(x,y_j)\big)d\nu(x) = 0;$

(c) $|H_{k,j}(x,y)| \lesssim 2^{-J\varepsilon} \frac{1}{V_k(x)+V_k(y)}$;

(d) $|H_{k,j}(x,y) - H_{k,j}(x',y)| \lesssim 2^{-J\varepsilon}(2^k\rho(x,x'))^{\varepsilon} \frac{1}{V_k(x)+V_k(y)}$,

where x' satisfies $\rho(x,x') \le 32A^3 2^{-k}$. Under the above conditions (a)–(d), using a similar argument to the proof of Lemma 2.1 and Lemma 2.3, we obtain that for all $k,\ell \in \mathbb{Z}$ and $x,y \in \mathbb{R}^{n+1}$,

$$\operatorname{supp}(D_\ell H_{k,j})(\cdot,y) \subset Q(y, 32A^3(2^{-\ell} \vee 2^{-k})); \tag{3.13}$$

$$\operatorname{supp}(D_\ell H_{k,j})(x,\cdot) \subset Q(x, 16A^2(2^{-\ell} \vee 2^{-k})); \tag{3.14}$$

$$|D_\ell H_{k,j}(x,y)| \lesssim 2^{-J\varepsilon} 2^{-|\ell-k|\varepsilon} \frac{1}{V_{\ell\wedge k}(x)+V_{\ell\wedge k}(y)}. \tag{3.15}$$

Set

$$H(x,y) = \sum_{j=1}^{N_J} (D_\ell H_{k,j})(x,y).$$

By (3.14), (3.15) and doubling condition on measure ν,

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} |H(x,y)|d\nu(y) &\le \sum_{j=1}^{N_J} \int_{Q_j \cap Q(x,16A^2(2^{-\ell}\vee 2^{-k}))} |(D_\ell H_{k,j})(x,y)|d\nu(y) \\ &\lesssim 2^{-J\varepsilon}2^{-|\ell-k|\varepsilon} \sum_{j=1}^{N_J} \int_{Q_j \cap Q(x,16A^2(2^{-\ell}\vee 2^{-k}))} \frac{d\nu(y)}{V_{\ell\wedge k}(x)+V_{\ell\wedge k}(y)} \\ &\lesssim 2^{-J\varepsilon}2^{-|\ell-k|\varepsilon}.\end{aligned}$$

Similarly, (3.13) and (3.15) yield

$$\int_{\mathbb{R}^{n+1}} |H(x,y)|d\nu(x) \lesssim 2^{-J\varepsilon}2^{-|\ell-k|\varepsilon}.$$

The above two inequalities imply

$$\|D_\ell(g^1_{k,M,J})\|_{L^p_\nu} \lesssim 2^{-J\varepsilon}2^{-|\ell-k|\varepsilon}\|g\|_{L^p_\nu}, \qquad 1 \le p \le \infty,$$

which shows that, for $1 \le q < \infty$,

$$\begin{aligned}\|g^1_{k,M,J}\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} &\lesssim 2^{-J\varepsilon}\left\{\sum_{\ell\in\mathbb{Z}} 2^{\ell\alpha q}2^{-|\ell-k|\varepsilon q}\right\}^{\frac{1}{q}} \|g\|_{L^p_\nu} \\ &\lesssim 2^{-J\varepsilon}2^{k\alpha}\|g\|_{L^p_\nu} \\ &\to 0 \qquad \text{as } J \to \infty.\end{aligned} \tag{3.16}$$

For $q = \infty$, the fact $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}} \subset \dot{\mathcal{B}}^{\alpha,\infty}_{p,\mathcal{P}}$ shows

$$\|g^1_{k,M,J}\|_{\dot{\mathcal{B}}^{\alpha,\infty}_{p,\mathcal{P}}} \to 0 \qquad \text{as } J \to \infty. \tag{3.17}$$

By (3.12), (3.16), (3.17) and Lemma 2.5,

$$\langle f, D_k g\rangle = \lim_{M\to\infty}\lim_{J\to\infty} \langle f, g^2_{k,M,J}\rangle = \lim_{M\to\infty}\lim_{J\to\infty} \sum_{j=1}^{N_J} D_k(f)(y_j) \int_{Q_j} g(y)d\nu(y), \tag{3.18}$$

where we use Lemma 2.1 (i) to know $D_k(x,y)=D_k(y,x)$. We now write

$$\sum_{j=1}^{N_J} D_k(f)(y_j)\int_{Q_j} g(y)d\nu(y) = \sum_{j=1}^{N_J}\int_{Q_j} D_k(f)(y)g(y)d\nu(y) + \int_{\mathbb{R}^{n+1}}\sum_{j=1}^{N_J}\Big\{\big(D_k(f)(y_j)-D_k(f)(y)\big)\chi_{Q_j}(y)\Big\}g(y)d\nu(y).$$

Notice that

$$\big|\big(D_k(y_j,x)-D_k(y,x)\big)\chi_{Q_j}(y)\big| = \big|\big(D_k(x,y_j)-D_k(x,y)\big)\chi_{Q_j}(y)\big| = |H_{k,j}(x,y)|$$

and

$$\begin{aligned}\|H_{k,j}(\cdot,y)\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} &= \Big\{\sum_{s\in\mathbb{Z}} 2^{s\alpha q}\|D_sH_{k,j}(\cdot,y)\|^q_{L^p_\nu}\Big\}^{\frac{1}{q}}\\ &\lesssim 2^{-J\varepsilon}2^{k\alpha}\Big\{\sum_{s\in\mathbb{Z}} 2^{(s-k)\alpha q}2^{-|s-k|\varepsilon q}\Big\}^{\frac{1}{q}}(V_k(y))^{\frac{1}{p}-1}\\ &\lesssim 2^{-J\varepsilon}2^{k\alpha}(V_k(y))^{\frac{1}{p}-1}\\ &\to 0 \qquad \text{as } J\to\infty.\end{aligned}$$

Then, for $f\in(\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}})'$,

$$\begin{aligned}\big|\big(D_k(f)(y_j)-D_k(f)(y)\big)\chi_{Q_j}(y)\big| &= \Big|\int\big(D_k(y_j,x)-D_k(y,x)\big)\chi_{Q_j}(y)f(x)d\nu(x)\Big|\\ &\le \big\|\big(D_k(y_j,\cdot)-D_k(y,\cdot)\big)\chi_{Q_j}\big\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}\|f\|_{(\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}})'}\\ &\to 0 \qquad \text{as } J\to\infty.\end{aligned}$$

The Lebesgue dominated convergence theorem shows that

$$\lim_{J\to\infty}\int_{\mathbb{R}^{n+1}}\sum_{j=1}^{N_J}\Big\{\big(D_k(f)(y_j)-D_k(f)(y)\big)\chi_{Q_j}(y)\Big\}g(y)d\nu(y)=0,$$

which together with (3.18) shows

$$\langle f, D_k(g)\rangle = \lim_{M\to\infty}\lim_{J\to\infty}\sum_{j=1}^{N_J}\int_{Q_j} D_k(f)(y)g(y)d\nu(y) = \int_{\mathbb{R}^{n+1}} D_k(f)(y)g(y)d\nu(y) = \langle D_k(f), g\rangle.$$

For the proof of **step 3**, given $g\in\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$, let

$$\widetilde{g}_{k,M}(x) = \int_{Q(0,M)} D_k^N(x,y)D_kT_N^{-1}(g)(y)d\nu(y), \qquad M>0.$$

Then $\widetilde{g}_{k,M}\in L^p_\nu\cap\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$. It follows from Theorem 1.3 that

$$\Big\|g-\sum_{|k|\le M}\widetilde{g}_{k,M}\Big\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} = \Big\|g-\sum_{|k|\le M} D_k^N D_kT_N^{-1}(g)\chi_{Q(0,M)}\Big\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} \to 0 \qquad \text{as } M\to\infty.$$

Hence, claim (3.10) is proved, and the proof of Theorem 1.5 is completed. $\square$

4. Besov spaces associated with sections

We now apply the Calderón-type reproducing formula (1.11) in L^2_ν to prove that the definition of $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ is independent of the choice of approximations to the identity.

Theorem 4.1. *Let $|\alpha| < \frac{\varepsilon}{4}$ and $1 \le p, q \le \infty$. Suppose that $\{S_k\}_{k\in\mathbb{Z}}$ and $\{P_k\}_{k\in\mathbb{Z}}$ are approximations to the identity associated with $\mathcal{P}$ on $(\mathbb{R}^{n+1}, \rho, \nu)$. Set $D_k = S_k - S_{k-1}$ and $E_k = P_k - P_{k-1}$. Then, for $f \in L^2_\nu$,*

$$\left\{\sum_{k\in\mathbb{Z}}\left(2^{k\alpha}\|D_k(f)\|_{L^p_\nu}\right)^q\right\}^{\frac{1}{q}} \approx \left\{\sum_{k'\in\mathbb{Z}}\left(2^{k'\alpha}\|E_{k'}(f)\|_{L^p_\nu}\right)^q\right\}^{\frac{1}{q}} \qquad \text{if } 1 \le q < \infty;$$

$$\sup_{k\in\mathbb{Z}} 2^{k\alpha}\|D_k(f)\|_{L^p_\nu} \approx \sup_{k'\in\mathbb{Z}} 2^{k'\alpha}\|E_{k'}(f)\|_{L^p_\nu} \qquad \text{if } q = \infty.$$

Proof. For $f \in L^2_\nu$, we have $f = \sum_{k'\in\mathbb{Z}} E^N_{k'} E_{k'} T_N^{-1}(f)$ in L^2_ν. Hence, there exists a subsequence (we write the same indices for simplicity) such that $f = \sum_{k'\in\mathbb{Z}} E^N_{k'} E_{k'} T_N^{-1}(f)$ almost everywhere. Then

$$D_k(f) = \sum_{k'\in\mathbb{Z}} D_k E^N_{k'} E_{k'} T_N^{-1}(f),$$

and Lemma 2.4 yields

$$\begin{aligned}\|D_k(f)\|_{L^p_\nu} &\le \sum_{k'\in\mathbb{Z}} \|D_k E^N_{k'}\|_{L^p_\nu \mapsto L^p_\nu} \|E_{k'} T_N^{-1}(f)\|_{L^p_\nu} \\ &\lesssim \sum_{k'\in\mathbb{Z}} \sum_{|s|\le N} 2^{-|k-k'-s|\varepsilon} \|E_{k'} T_N^{-1}(f)\|_{L^p_\nu}.\end{aligned} \tag{4.1}$$

For $1 \le q < \infty$, Hölder's inequality, (3.7) and (4.1) show that

$$\begin{aligned}&\left\{\sum_{k\in\mathbb{Z}}\left(2^{k\alpha}\|D_k(f)\|_{L^p_\nu}\right)^q\right\}^{\frac{1}{q}} \\ &\quad\lesssim \left\{\sum_{k\in\mathbb{Z}}\left(\sum_{k'\in\mathbb{Z}}\sum_{|s|\le N} 2^{-|k-k'-s|\varepsilon+(k-k'-s)\alpha} 2^{s\alpha}\right)^{\frac{q}{q'}}\right. \\ &\qquad\quad \left. \times \left(\sum_{k'\in\mathbb{Z}}\sum_{|s|\le N} 2^{-|k-k'-s|\varepsilon+(k-k'-s)\alpha} 2^{s\alpha} 2^{k'\alpha q} \|E_{k'} T_N^{-1}(f)\|^q_{L^p_\nu}\right)\right\}^{\frac{1}{q}} \\ &\quad\lesssim \left(\sum_{|s|\le N} 2^{s\alpha}\right)\left\{\sum_{k'\in\mathbb{Z}} 2^{k'\alpha q}\|E_{k'} T_N^{-1}(f)\|^q_{L^p_\nu}\right\}^{\frac{1}{q}} \\ &\quad\lesssim N 2^{N|\alpha|} \|T_N^{-1}(f)\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}} \\ &\quad\lesssim N 2^{N|\alpha|} \lambda_N \left\{\sum_{k'\in\mathbb{Z}}\left(2^{k'\alpha}\|E_{k'}(f)\|_{L^p_\nu}\right)^q\right\}^{\frac{1}{q}}.\end{aligned}$$

While $q=\infty$, using (3.7) and (4.1) again, we get

$$\begin{aligned}2^{k\alpha}\|D_k(f)\|_{L^p_\nu} &\lesssim \sum_{k'\in\mathbb{Z}}\sum_{|s|\le N} 2^{-|k-k'-s|\varepsilon+(k-k'-s)\alpha}2^{s\alpha}2^{k'\alpha}\|E_{k'}T_N^{-1}(f)\|_{L^p_\nu}\\ &\lesssim N2^{N|\alpha|}\|T_N^{-1}(f)\|_{\dot{\mathcal{B}}^{\alpha,\infty}_{p,\mathcal{P}}}\\ &\lesssim N2^{N|\alpha|}\lambda_N\sup_{k'\in\mathbb{Z}}2^{k'\alpha}\|E_{k'}(f)\|_{L^p_\nu} \qquad \text{for all } k\in\mathbb{Z}.\end{aligned}$$

Similarly, we have the reverse inequalities. □

Lemma 4.2. *Let* $|\alpha|<\frac{\varepsilon}{4}$ *and* $1\le p,q\le\infty$. *If* $f\in\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$, *then* $f\in\big(\dot{\mathcal{B}}^{-\alpha,q'}_{p',\mathcal{P}}\big)'$ *and*

$$\|f\|_{(\dot{\mathcal{B}}^{-\alpha,q'}_{p',\mathcal{P}})'}\lesssim N2^{N|\alpha|}\lambda_N\|f\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}.$$

Proof. Let $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation to the identity associated with $\mathcal{P}$ on $(\mathbb{R}^{n+1},\rho,\nu)$ and $D_k=S_k-S_{k-1}$. Given $f\in\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ and $g\in\dot{\mathcal{B}}^{-\alpha,q'}_{p',\mathcal{P}}$, Theorem 1.3 gives $f=\sum_{k\in\mathbb{Z}}D_k^ND_kT_N^{-1}(f)$ and Hölder's inequality shows

$$\begin{aligned}|\langle f,g\rangle| &= \left|\int_{\mathbb{R}^{n+1}}\sum_{k\in\mathbb{Z}}D_kT_N^{-1}(f)(z)D_k^N(g)(z)d\nu(z)\right|\\ &\le \sum_{k\in\mathbb{Z}}\|D_kT_N^{-1}(f)\|_{L^p_\nu}\|D_k^N(g)\|_{L^{p'}_\nu}\\ &\le\left\{\sum_{k\in\mathbb{Z}}2^{k\alpha q}\|D_kT_N^{-1}(f)\|^q_{L^p_\nu}\right\}^{\frac1q}\left\{\sum_{k\in\mathbb{Z}}2^{-k\alpha q'}\|D_k^N(g)\|^{q'}_{L^{p'}_\nu}\right\}^{\frac1{q'}}.\end{aligned}$$

Since $D_k^N=\sum_{|j|\le N}D_{k+j}$, we have

$$\begin{aligned}\left\{\sum_{k\in\mathbb{Z}}2^{-k\alpha q'}\|D_k^N(g)\|^{q'}_{L^{p'}_\nu}\right\}^{\frac1{q'}} &\le\left\{\sum_{k\in\mathbb{Z}}2^{-k\alpha q'}\Big(\sum_{|j|\le N}\|D_{k+j}(g)\|_{L^{p'}_\nu}\Big)^{q'}\right\}^{\frac1{q'}}\\ &\le\sum_{|j|\le N}2^{j\alpha}\left\{\sum_{k\in\mathbb{Z}}2^{-(k+j)\alpha q'}\|D_{k+j}(g)\|^{q'}_{L^{p'}_\nu}\right\}^{\frac1{q'}}\\ &\lesssim N2^{N|\alpha|}\|g\|_{\dot{\mathcal{B}}^{-\alpha,q'}_{p',\mathcal{P}}}.\end{aligned}$$

Thus, we apply (3.7) to obtain

$$|\langle f,g\rangle|\lesssim N2^{N|\alpha|}\lambda_N\|f\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}\|g\|_{\dot{\mathcal{B}}^{-\alpha,q'}_{p',\mathcal{P}}} \tag{4.2}$$

and hence

$$\|f\|_{(\dot{\mathcal{B}}^{-\alpha,q'}_{p',\mathcal{P}})'}\lesssim N2^{N|\alpha|}\lambda_N\|f\|_{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}},$$

which completes the proof. □

We now are ready to show Theorem 1.7.

Proof of Theorem 1.7. To prove $\dot{B}^{\alpha,q}_{p,\mathcal{P}} \subset \overline{\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}}$, given $f \in \dot{B}^{\alpha,q}_{p,\mathcal{P}}$, we will find a sequence of functions in $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ such that this sequence converges to f in $\dot{B}^{\alpha,q}_{p,\mathcal{P}}$. Lemma 2.5 and Theorem 1.5 yield

$$\begin{aligned} D_k(f)(x) &= \langle D_k(x,\cdot), f\rangle \\ &= \left\langle D_k(x,\cdot), \sum_{k'\in\mathbb{Z}} T_N^{-1} D_{k'} D_{k'}^N(f)\right\rangle \\ &= D_k\bigg(\sum_{k'\in\mathbb{Z}} T_N^{-1} D_{k'} D_{k'}^N(f)\bigg)(x) \qquad \text{almost everywhere,} \end{aligned}$$

which implies

$$\begin{aligned} D_k R_N(f)(z) &= D_k \sum_{k_0\in\mathbb{Z}} \sum_{|\ell_0|>N} D_{k_0+\ell_0} D_{k_0}(f)(z) \\ &= D_k \sum_{k_0\in\mathbb{Z}} \sum_{|\ell_0|>N} D_{k_0+\ell_0} D_{k_0}\bigg(\sum_{k'\in\mathbb{Z}} T_N^{-1} D_{k'} D_{k'}^N(f)\bigg)(z). \end{aligned}$$

Thus,

$$\begin{aligned} &D_k R_N(f)(z) \\ &= \sum_{k'\in\mathbb{Z}} \sum_{m=0}^{\infty} D_k\bigg(\sum_{k_0\in\mathbb{Z}} \sum_{|\ell_0|>N} D_{k_0+\ell_0} D_{k_0}\bigg) \cdots \bigg(\sum_{k_m\in\mathbb{Z}} \sum_{|\ell_m|>N} D_{k_m+\ell_m} D_{k_m}\bigg) D_{k'} D_{k'}^N(f)(z) \\ &= \sum_{k'\in\mathbb{Z}} \sum_{m=0}^{\infty} \sum_{k_0\in\mathbb{Z}} \sum_{|\ell_0|>N} \cdots \sum_{k_m\in\mathbb{Z}} \sum_{|\ell_m|>N} D_k D_{k_0+\ell_0} D_{k_0} \cdots D_{k_m+\ell_m} D_{k_m} D_{k'} D_{k'}^N(f)(z). \end{aligned}$$

Since the norms of $\dot{\mathcal{B}}^{\alpha,q}_{p,\mathcal{P}}$ and $\dot{B}^{\alpha,q}_{p,\mathcal{P}}$ are the same, the same argument as the proof of (3.1) shows

$$\|R_N(f)\|_{\dot{B}^{\alpha,q}_{p,\mathcal{P}}} \le C_0 N^{\frac{3}{2}} 2^{-N(\frac{\varepsilon}{4}-|\alpha|)} \|f\|_{\dot{B}^{\alpha,q}_{p,\mathcal{P}}}.$$

Hence T_N is bounded on $\dot{B}^{\alpha,q}_{p,\mathcal{P}}$ as well, so $T_N(g) = \sum_k D_k D_k^N(g)$ belongs to $\dot{B}^{\alpha,q}_{p,\mathcal{P}}$ provided $g \in \dot{B}^{\alpha,q}_{p,\mathcal{P}}$. Using the fact that $T_N^{-1} = (I - R_N)^{-1} = \sum_{m=0}^{\infty} (R_N)^m$, we obtain

$$\|T_N^{-1}(f)\|_{\dot{B}^{\alpha,q}_{p,\mathcal{P}}} \le \lambda_N \|f\|_{\dot{B}^{\alpha,q}_{p,\mathcal{P}}} \tag{4.3}$$

and then $\sum_k D_k D_k^N T_N^{-1}(f)$ belongs to $\dot{B}^{\alpha,q}_{p,\mathcal{P}}$. In order to prove that $\sum_k D_k D_k^N T_N^{-1}(f)$ converges to f in $\dot{B}^{\alpha,q}_{p,\mathcal{P}}$, we apply Lemma 2.5 and Theorem 1.5 again to get

$$D_k(f) = D_k\bigg(\sum_{k'\in\mathbb{Z}} D_{k'} D_{k'}^N T_N^{-1}(f)\bigg) \qquad \text{almost everywhere.}$$

Hence

$$D_k\bigg(f - \sum_{|k'|\le M} D_{k'} D_{k'}^N T_N^{-1}(f)\bigg) = D_k\bigg(\sum_{|k'|>M} D_{k'} D_{k'}^N T_N^{-1}(f)\bigg) \qquad \text{almost everywhere,}$$

and

$$\bigg\| f - \sum_{|k'|\le M} D_{k'} D_{k'}^N T_N^{-1}(f)\bigg\|_{\dot{B}^{\alpha,q}_{p,\mathcal{P}}} = \bigg\| \sum_{|k'|>M} D_{k'} D_{k'}^N T_N^{-1}(f)\bigg\|_{\dot{B}^{\alpha,q}_{p,\mathcal{P}}}.$$

The same argument as the proof of (3.8) gives

$$\Big\| f - \sum_{|k'|\le M} D_{k'} D_{k'}^N T_N^{-1}(f) \Big\|_{\dot B^{\alpha,q}_{p,\mathcal P}} \to 0 \qquad \text{as } M\to\infty. \tag{4.4}$$

Define $f_{k,M}$ by

$$f_{k,M}(z) = \int_{Q(0,M)} D_k(z,w)\big(D_k^N T_N^{-1}(f)\big)(w)d\nu(w), \qquad M>0.$$

Then $f_{k,M}\in \dot{\mathcal B}^{\alpha,q}_{p,\mathcal P}$ because of Remark 2.6. The above (4.4) gives

$$\Big\| f - \sum_{|k|\le M} f_{k,M} \Big\|_{\dot B^{\alpha,q}_{p,\mathcal P}} \to 0 \qquad \text{as } M\to\infty.$$

To show $\overline{\dot{\mathcal B}^{\alpha,q}_{p,\mathcal P}} \subset \dot B^{\alpha,q}_{p,\mathcal P}$, let $\{f_m\}_{m\in\mathbb N}\subset \dot{\mathcal B}^{\alpha,q}_{p,\mathcal P}$ be a Cauchy sequence with respect to the norm $\|\cdot\|_{\dot B^{\alpha,q}_{p,\mathcal P}}$. We will show that there is an $f\in \dot B^{\alpha,q}_{p,\mathcal P}$ such that f_m converges to f in $\dot B^{\alpha,q}_{p,\mathcal P}$ as $m\to\infty$. By Lemma 4.2,

$$\|f_n - f_m\|_{(\dot{\mathcal B}^{-\alpha,q'}_{p',\mathcal P})'} \lesssim N2^{N|\alpha|}\lambda_N \|f_n - f_m\|_{\dot{\mathcal B}^{\alpha,q}_{p,\mathcal P}},$$

which says that $\{f_m\}_{m\in\mathbb N}$ is also a Cauchy sequence in $\big(\dot{\mathcal B}^{-\alpha,q'}_{p',\mathcal P}\big)'$ with respect to the norm $\|\cdot\|_{(\dot{\mathcal B}^{-\alpha,q'}_{p',\mathcal P})'}$ and $\|f_m\|_{\dot B^{\alpha,q}_{p,\mathcal P}}\le C$. Since $\big(\dot{\mathcal B}^{-\alpha,q'}_{p',\mathcal P}\big)'$ is a Banach space (see [23, p. 111]), there exists an $f\in \big(\dot{\mathcal B}^{-\alpha,q'}_{p',\mathcal P}\big)'$ such that

$$\|f_m - f\|_{(\dot{\mathcal B}^{-\alpha,q'}_{p',\mathcal P})'} \to 0 \qquad \text{as } m\to\infty.$$

It follows from Lemma 2.5 that

$$|D_k(f_m - f)(z)| \le \|D_k(z,\cdot)\|_{\dot{\mathcal B}^{-\alpha,q'}_{p',\mathcal P}} \|f_m - f\|_{(\dot{\mathcal B}^{-\alpha,q'}_{p',\mathcal P})'},$$

which implies

$$\lim_{m\to\infty} D_k(f_m)(z) = D_k(f)(z). \tag{4.5}$$

By Fatou's lemma and (4.5),

$$\|f\|_{\dot B^{\alpha,q}_{p,\mathcal P}} \le \liminf_{m\to\infty} \|f_m\|_{\dot B^{\alpha,q}_{p,\mathcal P}} \le C,$$

which shows $f\in \dot B^{\alpha,q}_{p,\mathcal P}$. Applying Lebesgue dominated convergence theorem, we obtain that f_m converges to f in $\dot B^{\alpha,q}_{p,\mathcal P}$. □

To study the dual of $\dot B^{\alpha,q}_{p,\mathcal P}$, we need the following lemma.

Lemma 4.3. *Let $\{S_k\}_{k\in\mathbb Z}$ be an approximation to the identity associated with $\mathcal P$ on $(\mathbb R^{n+1},\rho,\nu)$ and $D_k = S_k - S_{k-1}$. For $|\alpha|<\frac{\varepsilon}{4}$ and $1\le p,q\le\infty$, if a sequence of functions $\{g_k\}_{k\in\mathbb Z}$ satisfies $\big\|\{2^{k\alpha}\|g_k\|_{L^p_\nu}\}_{k\in\mathbb Z}\big\|_{\ell^q}<\infty$, then $\sum_{k\in\mathbb Z} D_k(g_k)\in \dot B^{\alpha,q}_{p,\mathcal P}$ and*

$$\Big\|\sum_{k\in\mathbb Z} D_k(g_k)\Big\|_{\dot B^{\alpha,q}_{p,\mathcal P}} \lesssim \big\|\{2^{k\alpha}\|g_k\|_{L^p_\nu}\}_{k\in\mathbb Z}\big\|_{\ell^q}.$$

Proof. For $m_1, m_2 \in \mathbb{Z}$ with $m_1 < m_2$, define $g_{m_1}^{m_2} = \sum_{k=m_1}^{m_2} D_k(g_k)$. Given $f \in \dot{\mathcal{B}}_{p',\mathcal{P}}^{-\alpha,q'}$, Hölder's inequality yields

$$
\begin{aligned}
|\langle g_{m_1}^{m_2}, f\rangle| &\le \sum_{k=m_1}^{m_2} |\langle g_k, D_k(f)\rangle| \\
&\le \left\{ \sum_{k=m_1}^{m_2} \left(2^{k\alpha}\|g_k\|_{L^p_\nu}\right)^q \right\}^{\frac{1}{q}} \left\{ \sum_{k=m_1}^{m_2} \left(2^{-k\alpha}\|D_k(f)\|_{L^{p'}_\nu}\right)^{q'} \right\}^{\frac{1}{q'}} \\
&\le \left\{ \sum_{k=m_1}^{m_2} \left(2^{k\alpha}\|g_k\|_{L^p_\nu}\right)^q \right\}^{\frac{1}{q}} \|f\|_{\dot{\mathcal{B}}_{p',\mathcal{P}}^{-\alpha,q'}},
\end{aligned}
$$

which shows $g_{m_1}^{m_2} \in \big(\dot{\mathcal{B}}_{p',\mathcal{P}}^{-\alpha,q'}\big)'$ and

$$
\|g_{m_1}^{m_2}\|_{(\dot{\mathcal{B}}_{p',\mathcal{P}}^{-\alpha,q'})'} \le \left\{ \sum_{k=m_1}^{m_2} \left(2^{k\alpha}\|g_k\|_{L^p_\nu}\right)^q \right\}^{\frac{1}{q}}.
$$

If we set $g = \sum_{k\in\mathbb{Z}} D_k(g_k)$, then $g \in \big(\dot{\mathcal{B}}_{p',\mathcal{P}}^{-\alpha,q'}\big)'$ as well. Using Lemma 2.4 and Hölder's inequality, we get

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} \left(2^{j\alpha}\|D_j(g)\|_{L^p_\nu}\right)^q &\le \sum_{j\in\mathbb{Z}} \left(2^{j\alpha} \sum_{k\in\mathbb{Z}} \|D_j D_k(g_k)\|_{L^p_\nu}\right)^q \\
&\lesssim \sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} 2^{(j-k)\alpha - |j-k|\varepsilon} 2^{k\alpha}\|g_k\|_{L^p_\nu}\right)^q \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|g_k\|_{L^p_\nu}^q,
\end{aligned}
$$

which completes the proof. □

Proof of Theorem 1.8. (a) follows from Theorem 1.7 and (4.2). For (b), given a bounded linear functional $\mathcal{L}$ on $\dot{B}_{p,\mathcal{P}}^{\alpha,q}$, by Theorem 1.7 again, $\mathcal{L}$ is also a bounded linear functional on $\dot{\mathcal{B}}_{p,\mathcal{P}}^{\alpha,q}$ and

$$
|\mathcal{L}(f)| \le \|\mathcal{L}\|\|f\|_{\dot{\mathcal{B}}_{p,\mathcal{P}}^{\alpha,q}} \qquad \text{for } f \in \dot{\mathcal{B}}_{p,\mathcal{P}}^{\alpha,q}.
$$

Let $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation to the identity associated with $\mathcal{P}$ and set $D_k = S_k - S_{k-1}$. Then, for each $f \in \dot{\mathcal{B}}_{p,\mathcal{P}}^{\alpha,q}$, $\{D_k(f)\}_{k\in\mathbb{Z}}$ belongs to the sequence space

$$
\ell_q^\alpha(L^p_\nu) = \left\{ \{f_k\}_{k\in\mathbb{Z}} : \|\{f_k\}_{k\in\mathbb{Z}}\|_{\ell_q^\alpha(L^p_\nu)} := \Big(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|f_k\|_{L^p_\nu}^q\Big)^{\frac{1}{q}} < \infty \right\}.
$$

Define $\mathcal{L}_0$ on a subset of $\ell_q^\alpha(L^p_\nu)$ by

$$
\mathcal{L}_0\big(\{D_k(f)\}_{k\in\mathbb{Z}}\big) = \mathcal{L}(f) \qquad \text{for } f \in \dot{\mathcal{B}}_{p,\mathcal{P}}^{\alpha,q}.
$$

Hence,

$$
|\mathcal{L}_0\big(\{D_k(f)\}_{k\in\mathbb{Z}}\big)| \le \|\mathcal{L}\|\|f\|_{\dot{\mathcal{B}}_{p,\mathcal{P}}^{\alpha,q}} = \|\mathcal{L}\|\|\{D_k(f)\}_{k\in\mathbb{Z}}\|_{\ell_q^\alpha(L^p_\nu)}.
$$

The Hahn-Banach theorem shows that $\mathcal{L}_0$ can be extended to a functional $\overline{\mathcal{L}_0}$ on $\ell_q^\alpha(L_\nu^p)$. Since $(\ell_q^\alpha(L_\nu^p))' = \ell_{q'}^{-\alpha}(L_\nu^{p'})$ for $1 \le p, q < \infty$ (see [18, page 178]), there exists a unique sequence $\{g_k\}_{k\in\mathbb{Z}} \in \ell_{q'}^{-\alpha}(L_\nu^{p'})$ such that

$$\overline{\mathcal{L}_0}(\{f_k\}_{k\in\mathbb{Z}}) = \sum_{k\in\mathbb{Z}} \langle f_k, g_k\rangle \qquad \text{for all } \{f_k\}_{k\in\mathbb{Z}} \in \ell_q^\alpha(L_\nu^p)$$

and

$$\|\{g_k\}_{k\in\mathbb{Z}}\|_{\ell_{q'}^{-\alpha}(L_\nu^{p'})} \lesssim \|\overline{\mathcal{L}_0}\| \le \|\mathcal{L}\|.$$

For $f \in \dot{\mathcal{B}}_{p,\mathcal{P}}^{\alpha,q}$, we have

$$\mathcal{L}(f) = \mathcal{L}_0(\{D_k(f)\}_{k\in\mathbb{Z}}) = \sum_{k\in\mathbb{Z}} \langle D_k(f), g_k\rangle = \sum_{k\in\mathbb{Z}} \langle f, D_k(g_k)\rangle = \Big\langle f, \sum_{k\in\mathbb{Z}} D_k(g_k)\Big\rangle.$$

Let $g = \sum_{k\in\mathbb{Z}} D_k(g_k)$. Lemma 4.3 says that $g \in \dot{B}_{p',\mathcal{P}}^{-\alpha,q'}$ and

$$\|g\|_{\dot{B}_{p',\mathcal{P}}^{-\alpha,q'}} \lesssim \|\{g_k\}_{k\in\mathbb{Z}}\|_{\ell_{q'}^{-\alpha}(L_\nu^{p'})} \lesssim \|\mathcal{L}\|.$$

This completes the proof. □

5. Proof of the embedding theorem for $\dot{B}_{p,\mathcal{P}}^{\alpha,q}$

Proof of Theorem 1.10. Let p_1 and p_2 satisfy the assumption of Theorem 1.10. Set $\frac{1}{r} = \frac{1}{p_2} - \frac{1}{p_1}$, then $\frac{1}{p_2} + \frac{1}{r'} = 1 + \frac{1}{p_1}$. By Lemma 2.3,

$$\int |D_\ell D_k(x,y)|^{r'} d\nu(x) \lesssim \int_{Q(y,16A^2(2^{-\ell}\vee 2^{-k}))} 2^{-|k-\ell|\varepsilon r'} \Big(\frac{1}{V_{\ell\wedge k}(x) + V_{\ell\wedge k}(y)}\Big)^{r'} d\nu(x).$$

The doubling property and the lower bound condition (1.14) on the measure ν give

$$\int |D_\ell D_k(x,y)|^{r'} d\nu(x) \lesssim 2^{-|k-\ell|\varepsilon r'} \big(V_{\ell\wedge k}(y)\big)^{1-r'} \lesssim 2^{-|k-\ell|\varepsilon r'} (2^{-\ell\omega} \vee 2^{-k\omega})^{1-r'}.$$

Similarly,

$$\int |D_\ell D_k(x,y)|^{r'} d\nu(y) \lesssim 2^{-|k-\ell|\varepsilon r'} (2^{-\ell\omega} \vee 2^{-k\omega})^{1-r'}.$$

For $f \in \dot{B}_{p_2,\mathcal{P}}^{\alpha_2,q}$, Young's inequality yields

$$\begin{aligned}
&\|D_\ell D_k D_k^N T_N^{-1}(f)\|_{L_\nu^{p_1}} \\
&\lesssim \big(2^{-|k-\ell|\varepsilon r'}(2^{-\ell\omega}\vee 2^{-k\omega})^{1-r'}\big)^{\frac{1}{p_1}} \big(2^{-|k-\ell|\varepsilon r'}(2^{-\ell\omega}\vee 2^{-k\omega})^{1-r'}\big)^{1-\frac{1}{p_2}} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}} \\
&= \big(2^{-|k-\ell|\varepsilon r'}(2^{-\ell\omega}\vee 2^{-k\omega})^{1-r'}\big)^{\frac{1}{r'}} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}} \\
&= 2^{-|k-\ell|\varepsilon}(2^{-\ell\omega}\vee 2^{-k\omega})^{\frac{1}{p_1}-\frac{1}{p_2}} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}}.
\end{aligned} \tag{5.1}$$

When $1 \le q < \infty$, we use Theorem 1.5 to get

$$\|f\|_{\dot{B}_{p_1,\mathcal{P}}^{\alpha_1,q}} = \Big\{\sum_{\ell\in\mathbb{Z}} 2^{\alpha_1 \ell q} \Big\| D_\ell\Big(\sum_k D_k D_k^N T_N^{-1}(f)\Big)\Big\|_{L_\nu^{p_1}}^q \Big\}^{\frac{1}{q}}$$

$$\lesssim \Big\{ \sum_{\ell\in\mathbb{Z}} 2^{\alpha_1 \ell q} \Big(\sum_{k\in\mathbb{Z}} 2^{-|k-\ell|\varepsilon} (2^{-\ell\omega} \vee 2^{-k\omega})^{\frac{1}{p_1}-\frac{1}{p_2}} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}} \Big)^q \Big\}^{\frac{1}{q}}$$

$$\leq \Big\{ \sum_{\ell\in\mathbb{Z}} 2^{\alpha_1 \ell q} \Big(\sum_{k>\ell} 2^{-(k-\ell)\varepsilon} 2^{-\ell\omega(\frac{1}{p_1}-\frac{1}{p_2})} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}} \Big)^q \Big\}^{\frac{1}{q}}$$

$$+ \Big\{ \sum_{\ell\in\mathbb{Z}} 2^{\alpha_1 \ell q} \Big(\sum_{k\leq\ell} 2^{-(\ell-k)\varepsilon} 2^{-k\omega(\frac{1}{p_1}-\frac{1}{p_2})} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}} \Big)^q \Big\}^{\frac{1}{q}}$$

$$:= I + J.$$

For I, the condition $\frac{\omega}{p_1} - \frac{\omega}{p_2} = \alpha_1 - \alpha_2$ shows

$$I = \Big\{ \sum_{\ell\in\mathbb{Z}} 2^{\alpha_1 \ell q} \Big(\sum_{k>\ell} 2^{-(k-\ell)\varepsilon} 2^{-\ell(\alpha_1-\alpha_2)} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}} \Big)^q \Big\}^{\frac{1}{q}}$$

$$= \Big\{ \sum_{\ell\in\mathbb{Z}} \Big(\sum_{k>\ell} 2^{-(k-\ell)\varepsilon+(\ell-k)\alpha_2} 2^{k\alpha_2} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}} \Big)^q \Big\}^{\frac{1}{q}}.$$

Hölder's inequality and $|\alpha_2| < \frac{\varepsilon}{4}$ imply

$$I \lesssim \Big\{ \sum_{\ell\in\mathbb{Z}} \sum_{k>\ell} 2^{-(k-\ell)\varepsilon+(\ell-k)\alpha_2} 2^{k\alpha_2 q} \|D_k^N T_N^{-1}(f)\|^q_{L_\nu^{p_2}} \Big\}^{\frac{1}{q}}$$

$$= \Big\{ \sum_{k\in\mathbb{Z}} \Big(\sum_{\ell<k} 2^{-(k-\ell)(\varepsilon+\alpha_2)} \Big) 2^{k\alpha_2 q} \|D_k^N T_N^{-1}(f)\|^q_{L_\nu^{p_2}} \Big\}^{\frac{1}{q}}$$

$$\lesssim \Big\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha_2 q} \|D_k^N T_N^{-1}(f)\|^q_{L_\nu^{p_2}} \Big\}^{\frac{1}{q}}.$$

Minkowski's inequality and (4.3) give

$$I \lesssim \Big\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha_2 q} \Big(\sum_{|s|\leq N} \|D_{k+s} T_N^{-1}(f)\|_{L_\nu^{p_2}} \Big)^q \Big\}^{\frac{1}{q}}$$

$$\lesssim \sum_{|s|\leq N} 2^{-s\alpha_2} \Big\{ \sum_{k\in\mathbb{Z}} 2^{(k+s)\alpha_2 q} \|D_{k+s} T_N^{-1}(f)\|^q_{L_\nu^{p_2}} \Big\}^{\frac{1}{q}}$$

$$\lesssim N 2^{N|\alpha_2|} \|T_N^{-1}(f)\|_{\dot{B}^{\alpha_2,q}_{p_2,\mathcal{P}}}$$

$$\lesssim \frac{N 2^{N|\alpha_2|}}{1 - N^{\frac{3}{2}} 2^{-N(\frac{\varepsilon}{4}-|\alpha_2|)}} \|f\|_{\dot{B}^{\alpha_2,q}_{p_2,\mathcal{P}}}.$$

For J, we use the conditions $\frac{\omega}{p_1} - \frac{\omega}{p_2} = \alpha_1 - \alpha_2$, $|\alpha_1| < \frac{\varepsilon}{4}$ and Hölder's inequality to get

$$J = \Big\{ \sum_{\ell\in\mathbb{Z}} \Big(\sum_{k\leq\ell} 2^{-(\ell-k)\varepsilon+\alpha_1(\ell-k)} 2^{k\alpha_2} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}} \Big)^q \Big\}^{\frac{1}{q}}$$

$$\leq \Big\{ \sum_{\ell\in\mathbb{Z}} \Big(\sum_{k\leq\ell} 2^{-(\ell-k)\varepsilon+\alpha_1(\ell-k)} \Big)^{\frac{q}{q'}} \Big(\sum_{k\leq\ell} 2^{-(\ell-k)\varepsilon+\alpha_1(\ell-k)} 2^{k\alpha_2 q} \|D_k^N T_N^{-1}(f)\|^q_{L_\nu^{p_2}} \Big) \Big\}^{\frac{1}{q}}$$

$$\lesssim \left\{ \sum_{k\in\mathbb{Z}} \Big(\sum_{\ell\ge k} 2^{-(\ell-k)(\varepsilon-\alpha_1)} \Big) 2^{k\alpha_2 q} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}}^q \right\}^{\frac{1}{q}}$$

$$\lesssim \left\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha_2 q} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}}^q \right\}^{\frac{1}{q}}.$$

By Minkowski's inequality and (4.3) again,

$$\begin{aligned} J &\lesssim \sum_{|s|\le N} 2^{-s\alpha_2} \left\{ \sum_{k\in\mathbb{Z}} 2^{(k+s)\alpha_2 q} \|D_{k+s} T_N^{-1}(f)\|_{L_\nu^{p_2}}^q \right\}^{\frac{1}{q}} \\ &\lesssim N 2^{N|\alpha_2|} \|T_N^{-1}(f)\|_{\dot{B}^{\alpha_2,q}_{p_2,\mathcal{P}}} \\ &\lesssim \frac{N 2^{N|\alpha_2|}}{1 - N^{\frac{3}{2}} 2^{-N(\frac{\varepsilon}{4}-|\alpha_2|)}} \|f\|_{\dot{B}^{\alpha_2,q}_{p_2,\mathcal{P}}}. \end{aligned}$$

While $q=\infty$, it follows from Minkowski's inequality, Theorem 1.5, and (5.1) that

$$\begin{aligned} 2^{\alpha_1\ell}\|D_\ell(f)\|_{L_\nu^{p_1}} &\le 2^{\alpha_1\ell} \Big(\sum_{k\in\mathbb{Z}} \|D_\ell D_k D_k^N T_N^{-1}(f)\|_{L_\nu^{p_1}} \Big) \\ &\lesssim 2^{\alpha_1\ell} \Big(\sum_{k\in\mathbb{Z}} 2^{-|k-\ell|\varepsilon} (2^{-\ell\omega} \vee 2^{-k\omega})^{\frac{1}{p_1}-\frac{1}{p_2}} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}} \Big). \end{aligned}$$

Since $\frac{\omega}{p_1} - \frac{\omega}{p_2} = \alpha_1 - \alpha_2$ and $-\frac{\varepsilon}{4} < \alpha_1 < \alpha_2 < \frac{\varepsilon}{4}$,

$$\begin{aligned} 2^{\alpha_1\ell}\|D_\ell(f)\|_{L_\nu^{p_1}} &\lesssim 2^{\alpha_1\ell} \Big(\sum_{k>\ell} 2^{-(k-\ell)\varepsilon} 2^{-\ell(\alpha_1-\alpha_2)} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}} \Big) \\ &\quad + 2^{\alpha_1\ell} \Big(\sum_{k\le\ell} 2^{-(\ell-k)\varepsilon} 2^{-k(\alpha_1-\alpha_2)} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}} \Big) \\ &= \sum_{k>\ell} 2^{-(k-\ell)(\varepsilon+\alpha_2)} 2^{k\alpha_2} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}} \\ &\quad + \sum_{k\le\ell} 2^{-(\ell-k)(\varepsilon-\alpha_1)} 2^{k\alpha_2} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}} \\ &\lesssim \sup_{k\in\mathbb{Z}} 2^{k\alpha_2} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}}. \end{aligned}$$

Applying (4.3) again, we obtain

$$\begin{aligned} \|f\|_{\dot{B}^{\alpha_1,\infty}_{p_1,\mathcal{P}}} &= \sup_{\ell\in\mathbb{Z}} 2^{\alpha_1\ell} \|D_\ell(f)\|_{L_\nu^{p_1}} \lesssim \sup_{k\in\mathbb{Z}} 2^{k\alpha_2} \|D_k^N T_N^{-1}(f)\|_{L_\nu^{p_2}} \\ &\le \sum_{|s|\le N} 2^{-s\alpha_2} \sup_{k\in\mathbb{Z}} 2^{(k+s)\alpha_2} \|D_{k+s} T_N^{-1}(f)\|_{L_\nu^{p_2}} \\ &\lesssim N 2^{N|\alpha_2|} \|T_N^{-1}(f)\|_{\dot{B}^{\alpha_2,\infty}_{p_2,\mathcal{P}}} \\ &\lesssim \frac{N 2^{N|\alpha_2|}}{1 - N^{\frac{3}{2}} 2^{-N(\frac{\varepsilon}{4}-|\alpha_2|)}} \|f\|_{\dot{B}^{\alpha_2,\infty}_{p_2,\mathcal{P}}}, \end{aligned}$$

and the proof of Theorem 1.10 is completed. □

References

[1] L. A. Caffarelli and C. E. Gutiérrez, *Real analysis related to the Monge-Ampère equation*, Trans. Amer. Math. Soc. **348** (1996), 1075–1092.

[2] L. A. Caffarelli and C. E. Gutiérrez, *Properties of the solutions of the linearized Monge-Ampère equation*, Amer. J. Math. **119** (1997), 423–465.

[3] I. Chavel, *Riemannian Geometry: A Modern Introduction*, Cambridge University Press, 1993.

[4] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.

[5] T. Coulhon, G. Kerkyacharian, and P. Petrushev, *Heat kernel generated frames in the setting of Dirichlet spaces*, J. Fourier Anal. Appl. **18** (2012), 995–1066.

[6] G. David, J.-L. Journé, and S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation*, Rev. Mat. Iberoamericana **1** (1985), 1–56.

[7] P. Daskalopoulos and O. Savin, $C^{1,\alpha}$ *regularity of solutions to parabolic Monge-Ampère equations*, Amer. J. Math. **4** (2012), 1051–1087.

[8] C. E. Gutiérrez and Q. Huang, $W^{2,p}$ *estimates for the parabolic Monge-Ampère equation*, Arch. Ration. Mech. Anal. **159** (2001), 137–177.

[9] P. Hajlasz, P Koskela and H. Tuominen, *Sobolev embeddings, extensions and measure density condition*, J. Funct. Anal **254** (2008), 1217–1234.

[10] Y. Han, *The embedding theorem for the Besov and Trieble-Lizorkin spaces on spaces of homogeneous type*, Proc. Amer. Math. Soc. **123** (1995), 2181–2189.

[11] Q. Huang, *Harnack inequality for the linearized parabolic Monge-Ampère equation*, Trans. Amer. Math. Soc. **351** (1999), 2025–2054.

[12] N. V. Krylov, *Sequences of convex functions, and estimates of the maximum of the solution of a parabolic equation*, Sibirsk. Mat. Ž. **17** (1976), 290–303 (in Russian), English translation in Siberian Math. J. **17** (1976), 226–236.

[13] M.-Y. Lee, C.-C. Lin, and X.-F. Wu, *Characterization of Campanato spaces associated with parabolic sections*, Asian J. Math. **20** (2016), 183–198.

[14] R. A. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math. **33** (1979), 257–270.

[15] A. I. Nazarov and N. N. Ural'tseva, *Convex-monotone hulls and an estimate of the maximum of the solution of a parabolic equation*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Stesklov. **147** (1985), 95–109.

[16] L. Saloff-Coste, *Aspects of Soblev-type inequalities*, London Mathematical Society Lecture Note Series **289**, 2002.

[17] L. Tang, *Regularity results on the parabolic Monge-Ampère equation with VMO type data*, J. Differential Equations **255** (2013), 1646–1656.

[18] H. Triebel, *Theory of Function Spaces*, Birkhäuser Verlag, Basel, 1983.

[19] K. Tso, *Deforming a hypersurface by its Gauss-Kronecker curvature*, Comm. Pure Appl. Math. **38** (1985), 867–882.

[20] K. Tso, *On an Aleksandrov-Bakel'man type maximum principle for second-order parabolic equations*, Comm. Partial Differential Equations **10** (1985), 543–553.

[21] R. Wang and G. Wang, *On existence, uniqueness and regularity of viscosity solutions for the first initial-boundary value problems to parabolic Monge-Ampère equation*, Northeast. Math. J. **8** (1992), 417–446.

[22] R. Wang and G. Wang, *The geometric measure theoretical characterization of viscosity solutions to parabolic Monge-Ampère type equation*, J. Partial Differential Equations **6** (1993), 237–254.

[23] K. Yosida, *Functional Analysis*, sixth edition, Spring-Verlag, Berlin-New York, 1980.

Meifang Cheng & Meng Qu
School of Mathematical and Computer Sciences
Anhui Normal University
Wuhu 241003
China

Email: meifangcheng@126.com; qumeng@mail.ahnu.edu.cn

Chin-Cheng Lin
Department of Mathematics
National Central University
Chung-Li 320
Taiwan

and

National Center for Theoretical Sciences
1 Roosevelt Road, Sec. 4
National Taiwan University
Taipei 106
Taiwan

Email: clin@math.ncu.edu.tw