

# Abelian Varieties Over Finite Fields as Basic Abelian Varieties

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# ABELIAN VARIETIES OVER FINITE FIELDS AS BASIC ABELIAN VARIETIES

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ABSTRACT. In this note we show that any basic abelian variety with additional structures over an arbitrary algebraically closed field of characteristic  $p > 0$  is isogenous to another one defined over a finite field. We also show that the category of abelian varieties over finite fields up to isogeny can be embedded into the category of basic abelian varieties with suitable endomorphism structures. Using this connection, we derive a new mass formula for a finite orbit of polarized abelian surfaces over a finite field.

## 1. INTRODUCTION

In this note we work on abelian varieties over fields of characteristic  $p > 0$ , particularly on basic abelian varieties with additional structures (endomorphisms, a polarization and a level structure). Conceptually, an abelian variety with fixed additional structures is *basic* if the corresponding point in a moduli space of PEL-type over  $\overline{\mathbb{F}}_p$  lands in the minimal Newton stratum (Rapoport-Zink [6] and Rapoport [5]). The group-theoretic definition was introduced by Kottwitz [1]. This notion is geometric in the sense that an abelian variety with additional structures is basic if and only if its base change to any algebraically closed field extension is also basic. As isogenous abelian varieties land in the same Newton stratum, an abelian variety with additional structures that is isogenous to a basic one is also basic.

Let  $B$  be a finite-dimensional semi-simple  $\mathbb{Q}$ -algebra with a positive involution  $*$  and  $O_B$  an order in  $B$  stable under  $*$ . A polarized  $O_B$ -abelian variety is a triple  $(A, \lambda, \iota)$  where  $A$  is an abelian variety with polarization  $\lambda$  and  $\iota : O_B \rightarrow \text{End}(A)$  is a ring monomorphism which is compatible with  $\lambda$ . We recall the definition of basic polarized  $O_B$ -abelian varieties  $(A, \lambda, \iota)$  in Section 2.

Basic abelian varieties with additional structures share many similar properties with supersingular abelian varieties without additional structures, and many techniques employed there can be carried over here as well. For example, similar to supersingular abelian varieties, one can formulate a geometric mass for a finite orbit of basic abelian varieties and relate this geometric mass to an arithmetic mass defined by group theory. The well-known Deuring-Eichler mass formula is obtained in this fashion. We refer to [15] for more discussions in this aspect. In this paper we prove the following result, which may be regarded as another analogue property enjoyed by supersingular abelian varieties.

**Theorem 1.1.** *Let  $\underline{A} = (A, \lambda, \iota)$  be a basic polarized  $O_B$ -abelian variety over an algebraically closed field  $k$  of characteristic  $p > 0$ . Then there exists a polarized*

$O_B$ -abelian variety  $\underline{A}' = (A', \lambda', \iota')$  over a finite field  $\kappa$  and an  $O_B$ -linear isogeny  $\varphi : A' \otimes_{\kappa} k \rightarrow A$  over  $k$  that preserves the polarizations.

The second part of this note studies the converse to Theorem 1.1. We show that any abelian variety over a finite field can be regarded as a basic abelian variety with suitable endomorphism structures. More precisely, if  $A$  is an abelian variety over the finite field  $\mathbb{F}_q$  of  $q = p^s$  elements and  $F = \mathbb{Q}(\pi_A) \subset \text{End}(A) \otimes \mathbb{Q}$  is the  $\mathbb{Q}$ -subalgebra generated by its Frobenius endomorphism  $\pi_A$ , then the abelian variety  $A$  together with the  $F$ -action is a basic  $F$ -abelian variety (Proposition 4.1). See Remark 3.2 for the notion of a  $B$ -abelian variety being basic. A priori, the original definition of basic abelian varieties with additional structures requires both structures of endomorphisms and polarizations. However, similar to supersingular abelian varieties, polarizations play no role in the characterization of supersingularity.

Let  $\mathcal{A}_{\mathbb{F}_q}$  denote the category of abelian varieties over  $\mathbb{F}_q$  up to isogeny, and  $\mathcal{B}^{\text{rig}}$  be the category of basic abelian varieties with rigidified endomorphisms over  $\overline{\mathbb{F}}_p$  up to isogeny, defined in Section 4. We prove the following result.

**Theorem 1.2.** *There is a functor  $\Phi$  that embeds the category  $\mathcal{A}_{\mathbb{F}_q}$  as a full subcategory of  $\mathcal{B}^{\text{rig}}$ .*

Theorem 1.2 connects (polarized) abelian varieties over a finite field  $\mathbb{F}_q$  with basic (polarized)  $F$ -abelian varieties over  $\overline{\mathbb{F}}_p$  equipped with a suitable commutative semi-simple  $\mathbb{Q}$ -algebra  $F$ . This connection is particularly useful when the  $\mathbb{Q}$ -algebra  $F$  is fixed. In this case one may consider a smaller class of (polarized) abelian varieties over  $\mathbb{F}_q$  whose endomorphism rings contain the maximal order  $O_F$ . This smaller set of isomorphism classes of polarized abelian varieties over  $\mathbb{F}_q$  is embeddable into the basic locus of a moduli space of polarized  $O_F$ -abelian varieties; see Lemma 5.1 and (5.2). Below is an example where we use this embedding to derive a mass formula for a class of polarized abelian surfaces over  $\mathbb{F}_p$ .

Choose a simple abelian variety  $A_0$  over the prime finite field  $\mathbb{F}_p$  whose Frobenius endomorphism  $\pi_0$  satisfies that  $\pi_0^2 = p$ . Then  $A_0$  is a superspecial abelian surface, i.e. the base change  $A_0 \otimes \overline{\mathbb{F}}_p$  is isomorphic to the product of two supersingular elliptic curves. Let us consider the set  $\Lambda$  of isomorphism classes of principally polarized simple abelian surfaces  $(A, \lambda)$  over  $\mathbb{F}_p$  such that  $A$  is isogenous to  $A_0$ . Put  $F = \mathbb{Q}(\pi_0) = \mathbb{Q}(\sqrt{p})$  and  $O_F$  its ring of integers. Let  $\Lambda^{\text{max}} \subset \Lambda$  be the subset of classes  $[(A, \lambda)]$  such that  $O_F \subset \text{End}(A)$ . We can show that  $\Lambda^{\text{max}}$  is a nonempty set. As usual, the mass  $\text{Mass}(\Lambda^{\text{max}})$  of  $\Lambda^{\text{max}}$  is defined by

$$(1.1) \quad \text{Mass}(\Lambda^{\text{max}}) := \sum_{(A, \lambda) \in \Lambda^{\text{max}}} |\text{Aut}(A, \lambda)|^{-1}.$$

Then we show that  $\text{Mass}(\Lambda^{\text{max}})$  is equal to the mass of a finite Hecke orbit  $S$  in the superspecial locus of a Hilbert modular surface modulo  $p$ . Furthermore, using the geometric mass formula for the superspecial orbits established in [12], we obtain the mass formula

$$(1.2) \quad \text{Mass}(\Lambda^{\text{max}}) = \frac{\zeta_F(-1)}{4},$$

where  $\zeta_F(s)$  the Dedekind zeta function of  $F$  (see Section 5.2).

The paper is organized as follows. In Section 2 we recall the definition of basic abelian varieties with additional structures. The proof and some consequences of

Theorem 1.1 are given in Section 3. In Section 4 we show that any abelian variety over a finite field, together with the action of the center of its endomorphism algebra, is a basic abelian variety. This result is used to construct the functor  $\Phi$  in Theorem 1.2. In the last section we consider the isogeny class of simple supersingular abelian surfaces mentioned as above and compute the associated mass (1.1).

**Notation.** If  $M$  is a  $\mathbb{Z}$ -module or a  $\mathbb{Q}$ -module and  $\ell$  is a prime, we write  $M_\ell := M \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$  or  $M_\ell = M \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ , respectively. For any perfect field  $k$  of characteristic  $p > 0$ , denote by  $W(k)$  the ring of Witt vectors over  $k$ ,  $B(k)$  the field of fractions of  $W(k)$ ,  $\sigma$  the Frobenius map on  $W(k)$  and  $B(k)$  induced by  $\sigma : k \rightarrow k$ ,  $x \mapsto x^p$ . If  $F$  is a finite product of number fields  $F_i$ , denote by  $O_F$  the maximal order in  $F$ . A prime  $\mathfrak{p}$  of  $F$  over  $p$ , denoted by  $\mathfrak{p}|p$ , means a prime of  $F_i$  for some  $F_i$  or a prime ideal of  $O_F$  over  $p$ . For an abelian variety  $A$  over a field  $k$ , write  $\text{End}(A) = \text{End}_k(A)$  for the endomorphism ring of  $A$  over  $k$  and  $\text{End}^0(A) = \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$  for the endomorphism algebra of  $A$  over  $k$ . If  $A$  is defined over a finite field  $\mathbb{F}_q$ , we denote by  $\pi_A$  the Frobenius endomorphism of  $A$  over  $\mathbb{F}_q$ .

## 2. BASIC ABELIAN VARIETIES WITH ADDITIONAL STRUCTURES

In this section we recall the concept of basic abelian varieties with additional structures introduced by Kottwitz [1]. Our reference is Rapoport-Zink [6, p.11, p. 281 and 6.25, p. 291].

**2.1. Setting.** Let  $B$  be a finite-dimensional semi-simple algebra over  $\mathbb{Q}$  with a positive involution  $*$ , and  $O_B$  be an arbitrary order of  $B$  stable under  $*$ .

Recall that a non-degenerate  $\mathbb{Q}$ -valued skew-Hermitian  $B$ -space is a pair  $(V, \psi)$  where  $V$  is a left faithful finite  $B$ -module, and  $\psi : V \times V \rightarrow \mathbb{Q}$  is a non-degenerate alternating pairing such that  $\psi(bx, y) = \psi(x, b^*y)$  for all  $b \in B$  and all  $x, y \in V$ .

A *polarized  $O_B$ -abelian variety* (resp. *polarized  $B$ -abelian variety*) is a triple  $\underline{A} = (A, \lambda, \iota)$ , where  $(A, \lambda)$  is a polarized abelian variety and  $\iota : O_B \rightarrow \text{End}(A)$  (resp.  $\iota : B \rightarrow \text{End}^0(A)$ ) is a ring monomorphism such that  $\lambda \iota(b^*) = \iota(b)^t \lambda$  for all  $b \in O_B$ . Here  $\iota(b)^t : A^t \rightarrow A^t$  denotes the dual morphism of  $\iota(b)$ .

Let  $\underline{A}$  be a polarized  $O_B$ -abelian variety over  $k$ , where  $k$  is an arbitrary field. For any prime  $\ell$  (not necessarily invertible in  $k$ ), we write  $\underline{A}(\ell)$  for the associated  $\ell$ -divisible group with additional structures  $(A[\ell^\infty], \lambda_\ell, \iota_\ell)$ , where  $\lambda_\ell$  is the induced quasi-polarization from  $A[\ell^\infty]$  to  $A^t[\ell^\infty] = A[\ell^\infty]^t$  (the Serre dual), and  $\iota_\ell : (O_B)_\ell \rightarrow \text{End}(A[\ell^\infty])$  the induced ring monomorphism. If  $\ell \neq \text{char}(k)$ , let  $T_\ell(A)$  denote the  $\ell$ -adic Tate module of  $A$ ,  $V_\ell := T_\ell(A) \otimes \mathbb{Q}_\ell$ , and let

$$(2.1) \quad \rho_\ell : \mathcal{G}_k \rightarrow \text{GU}_{B_\ell}(V_\ell, e_\lambda)$$

be the associated Galois representation. Here  $\mathcal{G}_k := \text{Gal}(k_s/k)$  is the Galois group of  $k$ ,  $k_s$  a separably closure of  $k$ , and

$$\text{GU}_{B_\ell}(V_\ell, e_\lambda) := \{g \in \text{Aut}_{B_\ell}(V_\ell) \mid e_\lambda(gx, gy) = ce_\lambda(x, y) \text{ for some } c \in \mathbb{Q}_\ell^\times\}$$

is the group of  $B_\ell$ -linear similitudes with respect to the Weil pairing

$$e_\lambda = e_{\lambda, \ell} : T_\ell(A) \times T_\ell(A) \rightarrow \mathbb{Z}_\ell(1),$$

where

$$\mathbb{Z}_\ell(1) := \varprojlim \mu_{\ell^m}(k_s)$$

is the Tate twist.

If  $k$  is a perfect field of characteristic  $p$ , let  $M(\underline{A})$  denote the covariant Dieudonné module of  $\underline{A}$  with the additional structures and put  $N(\underline{A}) := M(\underline{A}) \otimes_{W(k)} B(k)$ , the rational Dieudonné module (or the isocrystal) with the additional structures; see [10, Section 1].

In this note we consider only the objects  $\underline{A} = (A, \lambda, \iota)$  for which there is a non-degenerate skew-Hermitian  $B$ -space  $(V, \psi)$  with  $2 \dim A = \dim_{\mathbb{Q}} V$ . Namely, we require that there exists a complex polarized  $O_B$ -abelian variety with the same dimension as  $A$ . For example, we exclude the case where  $A$  is a supersingular elliptic curve and  $B$  is the quaternion  $\mathbb{Q}$ -algebra ramified precisely at  $\{p, \infty\}$ .

**2.2. Basic abelian varieties.** Let  $k$  be any field of characteristic  $p$  and  $\bar{k}$  be an algebraic closure of  $k$ . Put  $W := W(\bar{k})$  and  $L := B(\bar{k})$ . Let  $(V_p, \psi_p)$  be a  $\mathbb{Q}_p$ -valued non-degenerate skew-Hermitian  $B_p$ -module. A polarized  $O_B$ -abelian variety  $\underline{A}$  over  $\bar{k}$  is said to be *related to*  $(V_p, \psi_p)$  if there is a  $(B_p \otimes_{\mathbb{Q}_p} L)$ -linear isomorphism  $\alpha : N(\underline{A}) \simeq (V_p, \psi_p) \otimes_{\mathbb{Q}_p} L$  which preserves the pairings for a suitable identification  $L(1) \simeq L$ .

Let  $G_p := \mathrm{GU}_{B_p}(V_p, \psi_p)$  be the algebraic group over  $\mathbb{Q}_p$  of  $B_p$ -linear similitudes with respect to the pairing  $\psi_p$ . A choice of  $\alpha$  gives rise to an element  $b \in G_p(L)$  by transport of structure of the Frobenius map on  $N(\underline{A})$ , that is,  $\alpha : N(\underline{A}) \simeq (V_p \otimes L, b(\mathrm{id} \otimes \sigma), \psi_p)$  becomes an isomorphism of isocrystals with additional structures. The  $\sigma$ -conjugacy class  $[b]$  of  $b$  in  $G_p(L)$  is independent of the choice of  $\alpha$ . The decomposition of  $V_p \otimes L$  into isotypic components (the components of a single slope) induces a  $\mathbb{Q}$ -graded structure, and thus defines a (slope) homomorphism  $\nu_b : \mathbf{D} \rightarrow G_p$  over some unramified finite extension  $\mathbb{Q}_p^s$  of  $\mathbb{Q}_p$ , where  $\mathbf{D}$  is the pro-torus over  $\mathbb{Q}_p$  with character group  $\mathbb{Q}$ . The set  $\nu_{[b]} = \{\nu_b\}$  for  $b \in [b]$  is the  $G_p(L)$ -conjugacy class of  $\nu_b$  for a single  $b \in [b]$ , called the Newton vector associated to  $N(\underline{A})$ .

**Definition 2.1.** (1) A polarized  $O_B$ -abelian variety  $\underline{A}$  over  $\bar{k}$  is said to be *basic with respect to*  $(V_p, \psi_p)$  if

- (a)  $\underline{A}$  is related to  $(V_p, \psi_p)$ , and
- (b) the slope homomorphism  $\nu_b : \mathbf{D} \rightarrow G_p$  for  $b \in [b]$  is central.

(2) The object  $\underline{A}$  over  $\bar{k}$  is said to be *basic* if it is basic with respect to  $(V_p, \psi_p)$  for some non-degenerate skew-Hermitian  $B_p$ -space  $(V_p, \psi_p)$ .

(3) A polarized  $O_B$ -abelian variety  $\underline{A}$  over any field  $k$  is said to be *basic* if its base change  $\underline{A} \otimes_k \bar{k}$  is basic.

Clearly a polarized  $O_B$ -abelian variety  $\underline{A}$  is basic if (and only if) it is so considered as polarized  $B$ -abelian variety. Two polarized  $B$ -abelian varieties  $\underline{A}_1$  and  $\underline{A}_2$  are said to be *isogenous*, denote  $\underline{A}_1 \sim \underline{A}_2$ , if there is a  $B$ -linear isogeny  $\varphi : A_1 \rightarrow A_2$  such that the pull-back  $\varphi^* \lambda_2$  is a  $\mathbb{Q}$ -multiple of  $\lambda_1$ . Clearly the property for an object  $\underline{A}$  being basic is an isogeny invariant property. From the definition it is also easy to see that this is a geometric notion: an object  $\underline{A} = (A, \lambda, \iota)$  over  $k$  is basic if and only if the base change  $\underline{A} \otimes_k k_1$  is basic for any algebraically closed field  $k_1 \supset k$ .

### 3. PROOF OF THEOREMS 1.1 AND ITS COROLLARIES

**3.1.** To prove Theorem 1.1, we need some properties of basic abelian varieties with additional structures. Let  $(V, \psi)$  be a non-degenerate ( $\mathbb{Q}$ -valued) skew-Hermitian

$B$ -space and let  $G := \mathrm{GU}_B(V, \psi)$  be the algebraic group over  $\mathbb{Q}$  of  $B$ -linear similitudes with respect to the pairing  $\psi$ .

Let  $F$  be the center of  $B$  and  $F_0$  be the  $\mathbb{Q}$ -subalgebra fixed by the induced involution on  $F$ , which we denote by  $a \mapsto \bar{a}$ . Let  $\Sigma_p$  be the set of primes  $\mathfrak{p}$  of  $F$  over  $p$ , and for each prime  $\mathfrak{p}|p$ , denote by  $\mathrm{ord}_{\mathfrak{p}}$  the corresponding  $p$ -adic valuation normalized in a way that  $\mathrm{ord}_{\mathfrak{p}}(p) = 1$ . Let  $F_p := F \otimes \mathbb{Q}_p = \prod_{\mathfrak{p}|p} F_{\mathfrak{p}}$  be the decomposition into a product of local fields. For each isocrystal  $N$  with an  $F_p$ -linear action, let

$$(3.1) \quad N = \bigoplus_{\mathfrak{p}|p} N_{\mathfrak{p}}$$

be the decomposition with respect to the  $F_p$ -action.

**Lemma 3.1** (Rapoport-Zink). *Let the notation be as above.*

(1) *The center  $Z$  of  $G$  is the algebraic subgroup over  $\mathbb{Q}$  whose group of  $R$ -points is*

$$Z(R) = \{x \in (F \otimes R)^{\times}; x\bar{x} \in R^{\times}\},$$

for any  $\mathbb{Q}$ -algebra  $R$ .

(2) *Let  $N$  be an isocrystal with additional structures and suppose that it is related to  $(V \otimes \mathbb{Q}_p, \psi)$ . Then  $N$  is basic with respect to  $(V \otimes \mathbb{Q}_p, \psi)$  if and only if each component  $N_{\mathfrak{p}}$  is isotypic. In particular, if  $N$  is basic, then  $N_{\mathfrak{p}}$  is supersingular for primes  $\mathfrak{p}$  with  $\mathfrak{p} = \bar{\mathfrak{p}}$ .*

PROOF. Statement (1) and the only if part of statement (2) are proved in 6.25 of [6]. The if part is easier: as each  $N_{\mathfrak{p}}$  is isotypic, say of slope  $r_{\mathfrak{p}}/s$ , the slope homomorphism  $s\nu_b$  factors through  $\mathbf{D} \rightarrow \mathbb{G}_m$  and the action of  $s\nu_b(p)$  on  $N_{\mathfrak{p}}$  is a scalar. Thus, the slope homomorphism  $\nu_b : \mathbf{D} \rightarrow G_p$  must be central. ■

*Remark 3.2.* Lemma 3.1 provides a simple criterion for checking a polarized  $B$ -abelian variety  $\underline{A} = (A, \lambda, \iota)$  being basic. Note that the assertion of the statement (2) depends only on the underlying structure of  $B$ -action, and not on the equipped polarization structure. Therefore, it makes sense to call a  $B$ -abelian variety  $(A, \iota)$  *basic* if for any  $B$ -linear polarization  $\lambda$  on  $(A, \iota)$ , the polarized  $B$ -abelian variety  $(A, \lambda, \iota)$  is basic in the sense of Definition 2.1. Such a polarization  $\lambda$  always exists; see Kottwitz [2, Lemma 9.2].

It follows from Lemma 3.1 that a  $B$ -abelian variety  $(A, \iota)$  is basic if and only if the  $F$ -abelian variety  $(A, \iota|_F)$  is basic, where  $\iota|_F$  is the restriction of  $\iota$  to  $F$ .

The following two lemmas are reorganized from [6, 6.26-6.29]; proofs are provided solely for the reader's convenience.

**Lemma 3.3.** *Given any set  $\{\lambda_{\mathfrak{p}}\}_{\mathfrak{p}|p}$  of rational numbers with  $0 \leq \lambda_{\mathfrak{p}} \leq 1$  and  $\lambda_{\mathfrak{p}} + \lambda_{\bar{\mathfrak{p}}} = 1$ , there is a positive integer  $s$  and  $u \in O_F[1/p]^{\times}$  such that*

$$(3.2) \quad u\bar{u} = q := p^s, \quad \text{and} \quad \mathrm{ord}_{\mathfrak{p}} u = s\lambda_{\mathfrak{p}}, \quad \forall \mathfrak{p} \in \Sigma_p.$$

PROOF. Consider the map

$$\mathrm{ord} : O_F \left[ \frac{1}{p} \right]^{\times} \rightarrow \bigoplus_{\mathfrak{p} \in \Sigma_p} (1/e_{\mathfrak{p}})\mathbb{Z}, \quad u \mapsto (\mathrm{ord}_{\mathfrak{p}}(u))_{\mathfrak{p} \in \Sigma_p},$$

where  $e_{\mathfrak{p}}$  is the ramification index of  $\mathfrak{p}$ . By Dirichlet's unit theorem, the image has rank  $|\Sigma_p|$  and is of finite index. Therefore, there are a positive integer  $s$  and an

element  $u \in O_F[1/p]^\times$  such that  $\text{ord}_{\mathbf{p}}(u) = s\lambda_{\mathbf{p}} =: r_{\mathbf{p}}$  for all  $\mathbf{p} \in \Sigma_p$ . Let  $q = p^s$  and  $u' := qu/\bar{u}$ , then one computes

$$\text{ord}_{\mathbf{p}} u' = 2r_{\mathbf{p}} \text{ and } u'\bar{u}' = q^2.$$

Replacing  $u$  by  $u'$  and  $q$  by  $q^2$ , one gets the desired result.  $\blacksquare$

The element  $u$  in Lemma 3.3 actually lies in  $O_F$  as  $\text{ord}_{\mathbf{p}}(u) \geq 0$  for all  $\mathbf{p}|p$ .

**Lemma 3.4.** *Fix  $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$  and  $q = p^s$  as in Lemma 3.3, and an positive integer  $g$ . Then there is a positive integer  $n$  such that for any basic  $g$ -dimensional polarized  $O_B$ -abelian variety  $\underline{A}$  over a finite extension  $\mathbb{F}_{q^m}$  of  $\mathbb{F}_q$  with slopes  $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$ , the  $n$ -th power of Frobenius morphism  $\pi_{\underline{A}}^n$  lies in  $\iota(F)$ .*

PROOF. We first prove that the statement holds for one such object  $\underline{A}$ , i.e. there is an integer  $n_A$  possibly depending on  $A$  such that  $\pi_{\underline{A}}^{n_A} \in \iota(F)$ . Clearly the statement depends only on the isogeny class of  $\underline{A}$ . Let  $M$  be the Dieudonné module of  $\underline{A}$ . Within the isogeny class, we can choose  $\underline{A}$  so that  $\iota(O_F) \subset \text{End}(A)$  and  $F^s M_{\mathbf{p}} = p^{r_{\mathbf{p}}} M_{\mathbf{p}}$  for all  $\mathbf{p} \in \Sigma_p$ , where  $r_{\mathbf{p}} = s\lambda_{\mathbf{p}}$  and  $M = \bigoplus_{\mathbf{p}|p} M_{\mathbf{p}}$  is the decomposition with respect to (3.1). Let  $u$  be as in Lemma 3.3, then  $\iota(u)^{-m} \pi_A$  is an automorphism of  $A$  that preserves the polarization as  $\iota(u)^{-m} \pi_A(M_{\mathbf{p}}) = M_{\mathbf{p}}$  for all  $\mathbf{p} \in \Sigma_p$ . Therefore, a power of this automorphism is the identity by a theorem of Serre. Thus, a power of  $\pi_A$  is contained in  $\iota(F)$ .

Let  $C := \text{End}_B^0(A)$ . As  $\dim C$  is bounded by  $4g^2$ , there is a fixed positive integer  $n$  such that  $\zeta^n = 1$  for any element  $\zeta \in C$  of finite order. By the result we just proved that  $\iota(u)^{-m} \pi_A \in C$  is of finite order, we have  $\pi_A^n \in \iota(F)$  for all such objects  $\underline{A}$ .  $\blacksquare$

**3.2. Proof of Theorem 1.1.** Let the notation be as in Theorem 1.1. It suffices to show that  $A$  has smCM, that is, any maximal commutative semi-simple  $\mathbb{Q}$ -subalgebra of  $\text{End}^0(A)$  has degree  $2 \dim A$ . Then by a theorem of Grothendieck (see a proof in [4] or [11]) there exists an abelian variety  $A'$  over a finite field  $\kappa$  and an isogeny  $\varphi : A' \otimes_{\kappa} k \rightarrow A$  over  $k$ . Replacing  $A'$  by one in its isogeny class if necessary, we may assume that  $A'$  admits an action  $\iota'$  of  $O_B$  so that the isogeny  $\varphi$  is  $O_B$ -linear. Take the pull-back polarization  $\lambda'$  on  $A'$ , which is clearly defined over a finite field extension of  $\kappa$ .

Let  $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$  be the set of slopes for  $\underline{A}$ . Take  $q = p^s$  and a positive integer  $n$  as in Lemmas 3.3 and 3.4. We can choose a field  $k_0$  finitely generated over  $\mathbb{F}_q$  over which  $\underline{A}$  is defined. The abelian variety  $\underline{A}$  extends to a polarized  $O_B$ -abelian scheme  $\underline{\mathbf{A}}$  over  $S = \text{Spec } R$  for a finitely generated  $\mathbb{F}_q$ -subalgebra  $R$  of  $k_0$  with fraction field  $\text{Frac}(R) = k_0$ . We may assume further that  $S$  is smooth over  $\text{Spec } \mathbb{F}_q$ . Let  $s$  be a closed point of  $S$  and  $\eta$  the generic point. By Grothendieck's specialization theorem, the special fiber  $\underline{\mathbf{A}}_s$  over  $s$  also has the same slopes  $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$ , and hence is basic.

We identify the endomorphism rings  $\text{End}_{k_0}(A) = \text{End}_R(\mathbf{A}) \subset \text{End}(\mathbf{A}_{\bar{s}})$ , and write  $\iota$  for the  $O_B$ -actions on these abelian varieties. Let

$$\rho_{\ell} : \pi_1(S, \bar{\eta}) \rightarrow \text{Aut}(T_{\ell}(A_{\bar{\eta}}))$$

be the associated  $\ell$ -adic representation. The action of  $\text{Gal}(\bar{\eta}/\eta)$  on  $T_{\ell}(A_{\bar{\eta}})$  factors through  $\rho_{\ell}$ . Again we identify the Tate modules  $T_{\ell}(\mathbf{A}_{\bar{s}}) = T_{\ell}(\mathbf{A}_{\tilde{S}_{\bar{s}}}) = T_{\ell}(A_{\bar{\eta}})$ , where  $\tilde{S}_{\bar{s}}$  is the (strict) Henselization of  $S$  at  $\bar{s}$ . Put  $V_{\ell}(A_{\bar{\eta}}) := T_{\ell}(A_{\bar{\eta}}) \otimes \mathbb{Q}_{\ell}$ .

Let  $\pi_{A_s}$  be the Frobenius morphism on  $\mathbf{A}_s$  and  $\text{Frob}_s$  the geometric Frobenius element in  $\pi_1(S, \bar{\eta})$  corresponding to the closed point  $s$ . We have

- (i)  $\pi_{A_s}^n \in \iota(F) \subset \text{End}(T_\ell(\mathbf{A}_s))$ , by Lemma 3.4;
- (ii)  $\rho_\ell(\text{Frob}_s^n) = \pi_{A_s}^n$  lies in the center  $Z(\mathbb{Q}_\ell)$  of  $\text{GU}_{B_\ell}(V_\ell(A_{\bar{\eta}}), \langle, \rangle)$ , by the identification of the Tate modules and (i);
- (iii) the Frobenius elements  $\text{Frob}_s$  for all closed points  $s$  generate a dense subgroup of  $\pi_1(S, \bar{\eta})$ .

Let  $G_\ell := \rho_\ell(\pi_1(S, \bar{\eta}))$  be the  $\ell$ -adic monodromy group. Let  $m_n : G_\ell \rightarrow G_\ell$  be the map  $x \mapsto x^n$ . It is an open mapping and its image contains an open subgroup  $U$  of  $G_\ell$ , which is of finite index. Clearly  $U$  lies in the center  $Z(\mathbb{Q}_\ell)$  by (ii) and (iii). Replacing  $k_0$  by a finite extension, we have  $G_\ell \subset Z(\mathbb{Q}_\ell)$ . Let  $\mathbb{Q}_\ell[\pi]$  be the (commutative) subalgebra of  $\text{End}(V_\ell(A_{\bar{\eta}}))$  generated by  $G_\ell$ . By Zarhin's theorem [16],  $\mathbb{Q}_\ell[\pi]$  is semi-simple and commutative, and  $\text{End}_{\mathbb{Q}_\ell[\pi]}(V_\ell(A_{\bar{\eta}})) = \text{End}(A) \otimes \mathbb{Q}_\ell$ . Hence any maximal commutative semi-simple  $\mathbb{Q}_\ell$ -subalgebra of  $\text{End}(A) \otimes \mathbb{Q}_\ell$  is also a maximal one in  $\text{End}(V_\ell(A_{\bar{\eta}}))$ . This shows that any maximal commutative semi-simple subalgebra of  $\text{End}^0(A)$  has degree  $2g$ , and hence completes the proof. ■

**3.3. Consequences.** In [15] we defined a class of polarized  $B$ -abelian varieties, called of arithmetic type. For these abelian varieties the “simple mass formula” in [15, Theorem 2.2] remain valid for algebraically closed ground fields, not just for finitely generated fields over a prime field. We related these  $B$ -abelian varieties with basic  $B$ -abelian varieties in the case where the ground field  $k$  is  $\overline{\mathbb{F}}_p$ ; see [15, Theorem 4.5]. Using Theorem 1.1, we extend this result to an arbitrary algebraically closed field  $k$  of characteristic  $p > 0$ ,

Recall that a polarized  $B$ -abelian variety  $(A, \lambda, \iota)$  over an algebraically closed field  $k$  of characteristic  $p > 0$  is said to be of *arithmetic type* if there is a model  $(A_0, \lambda_0, \iota_0)$  of  $(A, \lambda, \iota)$  over a subfield  $k_0$  finitely generated over  $\mathbb{F}_p$  such that the associated Galois representation  $\rho_\ell : \mathcal{G}_{k_0} \rightarrow \text{GU}_B(V_\ell(A_0), e_{\lambda, \ell})$  (Section 2.1) is central for some prime  $\ell \neq p$  (or equivalently, for all primes  $\ell \neq p$ , see [15, Proposition 3.10]). It is shown in [15, Section 3] that this is again a geometric notion which depends only on the underlying  $B$ -abelian variety  $(A, \iota)$  and not on the carried polarization structure  $\lambda$ .

**Theorem 3.5.** *A  $B$ -abelian variety  $(A, \iota)$  over an algebraically closed field  $k$  of characteristic  $p > 0$  is of arithmetic type if and only if it is basic.*

PROOF. By Theorem 1.1, there is a  $B$ -abelian variety  $(A_0, \iota_0)$  over  $\overline{\mathbb{F}}_p$  and a  $B$ -linear isogeny  $\varphi : (A_0, \iota_0) \otimes_{\overline{\mathbb{F}}_p} k \rightarrow (A, \iota)$ . As a result we can reduce the statement to the case where  $k = \overline{\mathbb{F}}_p$  and this is Theorem 4.5 of [15]. ■

**Proposition 3.6** (cf. [6, Corollary 6.29]). *Let  $K$  be a finite-dimensional semi-simple  $\mathbb{Q}$ -algebra that admits a positive involution. Let  $(A, \iota)$  and  $(A', \iota')$  be two basic  $K$ -abelian varieties over an algebraically closed field  $k$  of characteristic  $p > 0$ . Then we have*

$$(3.3) \quad \text{Hom}_K(A, A') \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \simeq \text{Hom}_K(V_\ell(A), V_\ell(A')) \quad \forall \ell \neq p,$$

and

$$(3.4) \quad \text{Hom}_K(A, A') \otimes_{\mathbb{Z}} \mathbb{Q}_p \simeq \text{Hom}_K((N, \mathcal{F}), (N', \mathcal{F})),$$



where  $N$  and  $N'$  are the isocrystals associated to  $(A, \iota)$  and  $(A', \iota')$ , respectively.

PROOF. Let  $E$  be the center of  $K$ . If (3.3) and (3.4) hold true where  $K$  is replaced by  $E$ , then (3.3) and (3.4) hold true. Note that  $A$  is a basic  $K$ -abelian variety if and only if it is a basic  $E$ -abelian variety (Remark 3.2). Replacing  $K$  by its center, we may assume that  $K$  is commutative.

By Theorem 1.1, there are  $K$ -abelian varieties  $(A_0, \iota_0)$  and  $(A'_0, \iota'_0)$  over  $\overline{\mathbb{F}}_p$  such that  $(A_0, \iota_0) \otimes_{\overline{\mathbb{F}}_p} k \sim (A, \iota)$  and  $(A'_0, \iota'_0) \otimes_{\overline{\mathbb{F}}_p} k \sim (A', \iota')$ . We have a natural isomorphism

$$\mathrm{Hom}_K(A_0, A'_0) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathrm{Hom}_K(A, A') \otimes_{\mathbb{Z}} \mathbb{Q},$$

and natural identifications  $V_\ell(A_0) = V_\ell(A)$  and  $V_\ell(A'_0) = V_\ell(A')$  for  $\ell \neq p$ . For  $\ell = p$ , we have also the identification  $\mathrm{Hom}_K((N_0, \mathcal{F}), (N'_0, \mathcal{F})) = \mathrm{Hom}_K((N, \mathcal{F}), (N', \mathcal{F}))$ , where  $N_0$  and  $N'_0$  are the isocrystals associated to  $(A_0, \iota_0)$  and  $(A'_0, \iota'_0)$ , respectively. Therefore, we are reduced to prove the statement where  $k = \overline{\mathbb{F}}_p$ , which is done by Rapoport-Zink (see [6, Corollary 6.29, p. 293]). ■

**Corollary 3.7.** *Let  $(A, \iota)$  and  $(A', \iota')$  be two basic  $B$ -abelian varieties over an algebraically closed field  $k$  of characteristic  $p$ , with slopes  $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$  and  $\{\lambda'_{\mathbf{p}}\}_{\mathbf{p}|p}$ , respectively. Then  $(A, \iota)$  and  $(A', \iota')$  are isogenous if and only if  $\lambda_{\mathbf{p}} = \lambda'_{\mathbf{p}}$  and  $\mathrm{rank} N_{\mathbf{p}} = \mathrm{rank} N'_{\mathbf{p}}$  for all  $\mathbf{p}|p$ .*

PROOF. This follows from Proposition 3.6. ■

#### 4. A CORRESPONDENCE

**4.1.** Let  $\mathbb{F}_q$  be the finite field of  $q = p^s$  elements. Let  $\mathcal{A}_{\mathbb{F}_q}$  denote the category of abelian varieties over  $\mathbb{F}_q$  up to isogeny. Let  $\mathcal{B}$  be the category defined as follows, which we call the category of basic abelian varieties with endomorphisms over  $\overline{\mathbb{F}}_p$  up to isogeny. The objects of  $\mathcal{B}$  consist of all triples  $(F, A, \iota)$ , where

- $F$  is a finite-dimensional commutative semi-simple  $\mathbb{Q}$ -algebra that admits a positive involution, and
- $(A, \iota)$  is a basic  $F$ -abelian variety over  $\overline{\mathbb{F}}_p$ .

For any two objects  $\underline{A}_1 = (F_1, A_1, \iota_1)$  and  $\underline{A}_2 = (F_2, A_2, \iota_2)$  in  $\mathcal{B}$ , a morphism in  $\mathrm{Hom}_{\mathcal{B}}(\underline{A}_1, \underline{A}_2)$  is a pair  $(\varphi, \tilde{\varphi})$ , where

- $\tilde{\varphi} : F_1 \rightarrow F_2$  is a  $\mathbb{Q}$ -linear algebra homomorphism in a broader sense that the image  $\tilde{\varphi}(1_{F_1})$  of the identity  $1_{F_1}$  may not be the identity  $1_{F_2}$ , and
- $\varphi$  is an element in  $\mathrm{Hom}(A_1, A_2) \otimes \mathbb{Q}$  which is  $(F_1, F_2)$ -equivariant in the sense that  $\varphi \circ \iota_1(a) = \iota_2(\tilde{\varphi}(a)) \circ \varphi$  for all  $a \in F_1$ .

Note that if the map  $\tilde{\varphi} : F_1 \rightarrow F_2$  as above is surjective, then  $\tilde{\varphi}(1_{F_1}) = 1_{F_2}$  (as  $\tilde{\varphi}(1_{F_1})y = y$  for all  $y \in F_2$ ), i.e. it is also a ring homomorphism. A reason we need to allow more general maps  $\tilde{\varphi}$  is as follows. Let  $(A_i, \iota_i)$  be an  $F_i$ -abelian variety for  $i = 1, 2$ , and  $\iota_1 \times \iota_2 : F_1 \times F_2 \rightarrow \mathrm{End}(A_1 \times A_2)$  the product map. Then the map  $\varphi = \mathrm{id}_{A_1} \times 0 : A_1 \rightarrow A_1 \times A_2$  is an  $(F_1, F_1 \times F_2)$ -equivariant with respect to the map  $\tilde{\varphi} = \mathrm{id}_{F_1} \times 0 : F_1 \rightarrow F_1 \times F_2$ . The latter map is not a ring homomorphism.

Clearly two objects  $\underline{A}_1$  and  $\underline{A}_2$  in  $\mathcal{B}$  are isomorphic if and only if there is a  $\mathbb{Q}$ -algebra isomorphism  $\tilde{\varphi} : F_1 \simeq F_2$ , and an  $(F_1, F_2)$ -equivariant quasi-isogeny  $\varphi : A_1 \rightarrow A_2$  over  $\overline{\mathbb{F}}_p$ .

The category  $\mathcal{B}$  is not yet good enough in comparison with the category of abelian varieties with fixed endomorphism structures; there are simply too many morphisms  $\tilde{\varphi}$  among the fields  $F$ . For example, when  $F_1 = F_2 = F$ , the usual notion of morphisms between two  $F$ -abelian varieties would require  $\tilde{\varphi}$  to be the identity and not an arbitrary automorphism as in the category  $\mathcal{B}$ .

We introduce another category  $\mathcal{B}^{\text{rig}}$ , which we call the category of basic abelian varieties with rigidified endomorphisms over  $\overline{\mathbb{F}}_p$  up to isogeny. The objects of  $\mathcal{B}^{\text{rig}}$  consist of all tuples  $(F, x, A, \iota)$  over  $\overline{\mathbb{F}}_p$ , where  $(F, A, \iota)$  is an object in  $\mathcal{B}$  and  $x \in F$  is an element generating  $F$  over  $\mathbb{Q}$ . Suppose  $(F, x, A, \iota)$  is an object in  $\mathcal{B}^{\text{rig}}$ , let  $\mathbb{Q}[t] \rightarrow F$  be the natural surjective map sending  $t$  to  $x$ , and  $f : \mathbb{Q}[t] \rightarrow \text{End}^0(A)$  be the morphism obtained by composing with the map  $\iota$ . Given two objects  $\underline{A}_i = (F_i, x_i, A_i, \iota_i)$  in  $\mathcal{B}^{\text{rig}}$  ( $i = 1, 2$ ), a morphism  $\varphi : \underline{A}_1 \rightarrow \underline{A}_2$  in  $\mathcal{B}^{\text{rig}}$  is an element  $\varphi \in \text{Hom}(A_1, A_2) \otimes \mathbb{Q}$  such that  $\varphi \circ f_1(a) = f_2(a) \circ \varphi$  for all  $a \in \mathbb{Q}[t]$ , where  $f_i : \mathbb{Q}[t] \rightarrow \text{End}^0(A_i)$  are the maps associated as above. In the case when  $F_1 = F_2 = F$ , we have

$$\text{Hom}_F((A_1, \iota_1), (A_2, \iota_2)) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{Hom}_{\mathcal{B}^{\text{rig}}}((F, x, A_1, \iota_1), (F, x, A_2, \iota_2))$$

for any element  $x$  generates  $F$  over  $\mathbb{Q}$ , which recovers the usual notion of morphisms of  $F$ -abelian varieties (though we may not really want the additional structure  $x$ ).

We shall embed  $\mathcal{A}_{\mathbb{F}_q}$  as a full subcategory of  $\mathcal{B}^{\text{rig}}$ . As the first step, we prove the following result.

**Proposition 4.1.** *Let  $A$  be an abelian variety over  $\mathbb{F}_q$  and  $\pi_A$  its Frobenius endomorphism. Put  $F := \mathbb{Q}(\pi_A)$  and  $\iota : F \rightarrow \text{End}^0(A)$  for the inclusion. Then the  $F$ -abelian variety  $(A, \iota)$  is basic.*

PROOF. Suppose that the finite field  $k$  has  $q = p^s$  elements. Let  $A \sim \prod_{i=1}^t A_i^{n_i}$  be the decomposition into components up to isogeny, where each abelian variety  $A_i$  is simple and  $A_i \not\sim A_j$  for any  $i \neq j$ . Let  $\pi_i$  be the Frobenius endomorphism of  $A_i$  and put  $F_i := \mathbb{Q}(\pi_i)$ . Then we have  $F = \prod_i F_i$ . Let  $\Sigma_{p,i}$  be the set of the primes  $\mathfrak{p}$  of  $F_i$  over  $p$ . Thus,  $\Sigma_p$  is the disjoint union of  $\Sigma_{p,i}$  for  $i = 1, \dots, t$ . Let  $N$  (resp.  $N_i$ ) be the isocrystal associated to the  $F$ -abelian variety  $\underline{A} = (A, \iota)$  (resp.  $\underline{A}_i = (A_i, \iota_i)$ ). Clearly if  $\mathfrak{p} \in \Sigma_{p,i}$  then  $N_{\mathfrak{p}} = N_{i,\mathfrak{p}}$ . In particular,  $N_{\mathfrak{p}}$  is isotypic for all  $\mathfrak{p} \in \Sigma_p$  if and only if  $N_{i,\mathfrak{p}}$  is isotypic for all  $i$  and all  $\mathfrak{p} \in \Sigma_{p,i}$ . It follows from Lemma 3.1 that  $\underline{A}$  is basic if and only if  $\underline{A}_i$  is basic for all  $i = 1, \dots, t$ . Therefore, it suffices to prove the statement when  $A$  is simple. In this case, as  $F^s = \pi$  and  $\pi \in F_{\mathfrak{p}}$ , the component  $N_{\mathfrak{p}}$  has slope  $\text{ord}_{\mathfrak{p}}(\pi)/s$ . ■

By Lemma 3.1, if  $K$  is any commutative semi-simple  $\mathbb{Q}$ -subalgebra of the endomorphism algebra  $\text{End}^0(A)$  which is stable under a Rosati involution and contains  $F$ , then  $(A, i)$  with  $i : K \subset \text{End}^0(A)$ , is also a basic  $K$ -abelian variety. Our way of making  $A$  into a basic abelian variety with endomorphism structures as in Proposition 4.1 is, after a suitable base change, the most “economical” one. Namely, one uses the least endomorphisms.

**Proposition 4.2.** *Let  $A$  be an abelian variety over  $\mathbb{F}_q$  such that  $\text{End}(A) = \text{End}(\overline{A})$ , where  $\overline{A} = A \otimes \overline{\mathbb{F}}_p$ . Suppose that  $K$  is a commutative semi-simple  $\mathbb{Q}$ -algebra admitting a positive involution, and  $(A, \iota)$  is a basic  $K$ -abelian variety. Then  $\iota(K)$  contains the center  $F$  of the endomorphism algebra  $\text{End}^0(A)$ .*

PROOF. Let  $\pi$  be the Frobenius endomorphism of  $A$ . Then for any positive integer  $n$  one has  $F = \mathbb{Q}(\pi^n)$  as  $F$  is the center of the endomorphism algebra  $\text{End}^0(A \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n})$ . Now using Lemma 3.4, there is a positive integer  $n$  such that  $\pi^n$  is contained in  $\iota(K)$ . As a result, the center  $F$  is contained in  $\iota(K)$ . ■

Now we define a functor  $\Phi : \mathcal{A}_{\mathbb{F}_q} \rightarrow \mathcal{B}^{\text{rig}}$  as follows. To each abelian variety  $A$  over  $\mathbb{F}_q$  we associate a tuple  $(F, \pi_A, \bar{A}, \iota)$ , where  $\pi_A$  is the Frobenius endomorphism of  $A$ ,  $F := \mathbb{Q}(\pi_A)$ ,  $\bar{A} := A \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_p$  and  $\iota : F \rightarrow \text{End}^0(\bar{A})$  is the inclusion. Clearly we have the associated map

$$(4.1) \quad \Phi_* : \text{Hom}(A_1, A_2) \otimes \mathbb{Q} \rightarrow \text{Hom}_{\mathcal{B}^{\text{rig}}}(\Phi(A_1), \Phi(A_2))$$

as  $\varphi \circ \iota_1(\pi_{A_1}) = \iota_2(\pi_{A_2}) \circ \varphi$  for any map  $\varphi \in \text{Hom}(A_1, A_2) \otimes \mathbb{Q}$ .

**Theorem 4.3.** *The functor  $\Phi : \mathcal{A}_{\mathbb{F}_q} \rightarrow \mathcal{B}^{\text{rig}}$  is fully faithful.*

PROOF. Let  $A_1$  and  $A_2$  be two abelian varieties over  $\mathbb{F}_q$ , and let  $\underline{A}_i := (F_i, \pi_i, \bar{A}_i, \iota_i)$  be the associated object in  $\mathcal{B}^{\text{rig}}$  for  $i = 1, 2$ . We must show that the associated map  $\Phi_*$  in (4.1) is bijective. It is clear that  $\Phi_*$  is injective. Let  $f : \bar{A}_1 \rightarrow \bar{A}_2$  be an element in  $\text{Hom}_{\mathcal{B}^{\text{rig}}}(\Phi(A_1), \Phi(A_2))$ , particularly  $\pi_2 \bar{f} = \bar{f} \pi_1$ . As  $\sigma_q(\bar{f}) = \pi_2 \bar{f} \pi_1^{-1} = \bar{f}$ , where  $\sigma_q \in \text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_q)$  is the Frobenius map, the morphism  $\bar{f}$  is defined over  $\mathbb{F}_q$ . ■

**4.2.** We restrict the functor  $\Phi$  to the objects for which the endomorphism algebras have a common center. Fix any abelian variety  $A_0$  over  $\mathbb{F}_q$ . Let  $\pi_0$  be the Frobenius endomorphism of  $A_0$  over  $\mathbb{F}_q$ ,  $p(t) \in \mathbb{Z}[t]$  its minimal polynomial over  $\mathbb{Q}$  and  $F := \mathbb{Q}[t]/(p(t))$ . A commutative semi-simple  $\mathbb{Q}$ -algebra  $F$  arising in this way is called a *q-Weil  $\mathbb{Q}$ -algebra*.

Let  $\mathcal{A}_{\pi_0, \mathbb{F}_q}$  denote the full subcategory of  $\mathcal{A}_{\mathbb{F}_q}$  consisting of all abelian varieties  $A$  such that the minimal polynomial of the Frobenius endomorphism of  $A$  is equal to  $p(t)$ . In other words, every abelian variety  $A$  over  $\mathbb{F}_q$  in  $\mathcal{A}_{\pi_0, \mathbb{F}_q}$  shares the same simple components of  $A_0$  up to isogeny.

Let  $\mathcal{B}_F$  denote the category of basic  $F$ -abelian varieties over  $\bar{\mathbb{F}}_p$  up to isogeny. Similarly we define a functor

$$(4.2) \quad \Phi_F : \mathcal{A}_{\pi_0, \mathbb{F}_q} \rightarrow \mathcal{B}_F, \quad A \mapsto (\bar{A}, \iota),$$

where  $\bar{A} := A \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_p$  and  $\iota : F \rightarrow \text{End}^0(\bar{A})$  is the ring monomorphism sending  $t$  to  $\pi_A$ . By Theorem 4.3, we obtain the following result.

**Proposition 4.4.** *For any q-Weil  $\mathbb{Q}$ -algebra  $F = \mathbb{Q}(\pi_0)$ , the functor  $\Phi_F : \mathcal{A}_{\pi_0, \mathbb{F}_q} \rightarrow \mathcal{B}_F$  is fully faithful.*

*Remark 4.5.* The functor  $\Phi_F$  is usually not essentially surjective. For example take  $q = p^2$  and  $\pi_0 = p\zeta_6$  with  $p \equiv 1 \pmod{3}$ . The corresponding abelian variety  $A_0$  is a simple supersingular abelian surface, and any object in  $\mathcal{A}_{\pi_0, \mathbb{F}_q}$  is isogenous to a finite product of copies of  $A_0$ . However, as  $F = \mathbb{Q}(\sqrt{-3})$  and  $p$  splits in  $F$ , there is an ordinary elliptic curve  $E$  over  $\bar{\mathbb{F}}_p$  and there is an isomorphism  $i : F \simeq \text{End}^0(E)$ . The  $F$ -elliptic curve  $(E, i)$  is clearly in  $\mathcal{B}_F$  but is not isogenous to a finite product of copies of  $A_0$ . In this case the functor  $\Phi_F$  is not essentially surjective. A point is that different Weil numbers can generate the same field.

## 5. A MASS FORMULA

**5.1. Within a simple isogeny class.** Let  $\pi$  be a Weil  $q$ -number,  $F = \mathbb{Q}(\pi)$  the number field generated by  $\pi$  over  $\mathbb{Q}$ , and  $O_F$  the ring of integers in  $F$ . Let  $\text{Isog}(\pi)$  denote the simple isogeny class corresponding to  $\pi$  by the Honda-Tate theory [8]. Let  $A_0$  be an abelian variety over  $\mathbb{F}_q$  in  $\text{Isog}(\pi)$  and put  $d := \dim(A_0)$ .

Let  $\Lambda(\pi)$  denote the set of isomorphism classes of abelian varieties over  $\mathbb{F}_q$  in  $\text{Isog}(\pi)$ , and  $\Lambda(\pi)^{\max} \subset \Lambda(\pi)$  be the subset consisting of all abelian varieties  $A$  such that the ring  $O_F$  is contained in  $\text{End}(A)$ . Let  $\mathbf{B}_{d,O_F}$  denote the set of isomorphism classes of  $d$ -dimensional basic  $O_F$ -abelian varieties over  $\overline{\mathbb{F}}_p$ .

The following lemma follows from Proposition 4.4.

**Lemma 5.1.** *The association  $A \mapsto (\overline{A}, \iota)$  induces an injective map  $\Phi_\pi : \Lambda(\pi)^{\max} \rightarrow \mathbf{B}_{d,O_F}$ .*

If  $A \in \Lambda(\pi)^{\max}$  is an abelian variety over  $\mathbb{F}_q$  and  $(\overline{A}, \iota)$  the corresponding basic  $O_F$ -abelian variety over  $\overline{\mathbb{F}}_p$ , then clearly any  $O_F$ -linear polarization  $\overline{\lambda}$  on  $(\overline{A}, \iota)$  descends uniquely to a polarization  $\lambda$  on  $A$  over  $\mathbb{F}_q$ . Particularly, the map  $\lambda \mapsto \overline{\lambda}$  gives rise to a one-to-one correspondence between polarizations on  $A$  and  $O_F$ -linear polarizations on  $(\overline{A}, \iota)$  over  $\overline{\mathbb{F}}_p$ . It follows that  $A$  admits a principal polarization if and only if  $(\overline{A}, \iota)$  admits a principal  $O_F$ -linear polarization. Moreover, we also have a natural isomorphism of finite groups

$$(5.1) \quad \text{Aut}(A, \lambda) \simeq \text{Aut}(\overline{A}, \overline{\lambda}, \iota).$$

Now we let  $\Lambda(\pi)_1^{\max}$  be the set of isomorphism classes of principally polarized abelian varieties  $(A, \lambda)$  over  $\mathbb{F}_q$  such that the underlying abelian variety  $A$  belongs to  $\Lambda(\pi)^{\max}$ . The set  $\Lambda(\pi)_1^{\max}$  could be empty; nevertheless, it is always finite. This follows from the finiteness of the set  $\mathcal{A}_{d,1}(\mathbb{F}_q)$  of  $\mathbb{F}_q$ -rational points of the Siegel modular variety  $\mathcal{A}_{d,1}$ ,

Let  $\mathcal{A}_{d,O_F,1}$  be the moduli space over  $\overline{\mathbb{F}}_p$  of  $d$ -dimensional principally polarized  $O_F$ -abelian varieties, and  $\mathbf{B}_{d,O_F,1} \subset \mathcal{A}_{d,O_F,1}(\overline{\mathbb{F}}_p)$  be its basic locus. Then the map  $\Phi_\pi$  induces an injective map

$$(5.2) \quad \Phi_\pi : \Lambda(\pi)_1^{\max} \rightarrow \mathbf{B}_{d,O_F,1}.$$

We have the following commutative diagram

$$\begin{array}{ccc} \Lambda(\pi)_1^{\max} & \xrightarrow{\Phi_\pi} & \mathbf{B}_{d,O_F,1} \\ \downarrow & & \downarrow \\ \Lambda(\pi)^{\max} & \xrightarrow{\Phi_\pi} & \mathbf{B}_{d,O_F}, \end{array}$$

where the vertical maps forget the polarization.

The mass of  $\Lambda(\pi)_1^{\max}$  is defined as

$$(5.3) \quad \text{Mass}(\Lambda(\pi)_1^{\max}) := \sum_{(A,\lambda) \in \Lambda(\pi)_1^{\max}} |\text{Aut}(A, \lambda)|^{-1}$$

if it is nonempty, and to be zero otherwise. Similarly, any finite subset  $S \subset \mathcal{A}_{d,O_F,1}(\overline{\mathbb{F}}_p)$ , the mass of  $S$  is defined as

$$(5.4) \quad \text{Mass}(S) := \sum_{(\overline{A}, \overline{\lambda}, \iota) \in S} |\text{Aut}(\overline{A}, \overline{\lambda}, \iota)|^{-1}$$

if  $S$  is nonempty and  $\text{Mass}(S) = 0$  otherwise. It follows from (5.1) that

$$(5.5) \quad \text{Mass}(\Lambda(\pi)_1^{\max}) = \text{Mass}(\text{Im}\Phi_\pi).$$

**5.2. An example with  $\pi = \sqrt{p}$ .** We consider a special case of the previous construction when  $\pi = \sqrt{p}$ . The result we obtain is the following.

**Theorem 5.2.** *Let  $\pi = \sqrt{p}$ . Then the finite set  $\Lambda(\pi)_1^{\max}$  is nonempty and we have*

$$(5.6) \quad \text{Mass}(\Lambda(\pi)_1^{\max}) = \frac{1}{4} \zeta_{\mathbb{Q}(\sqrt{p})}(-1).$$

We need a general result.

**Proposition 5.3.** *Let  $F$  be a totally real field,  $\mathcal{O} := O_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$  and  $k$  an algebraically closed field of characteristic  $p > 0$ .*

(1) *Let  $\underline{M} = (M, \langle, \rangle, \iota_M)$  be a supersingular separably quasi-polarized Dieudonné  $\mathcal{O}$ -module over  $k$  satisfying the following condition*

$$(*) \quad \text{tr}(\iota_M(a)) \cdot [F : \mathbb{Q}] = (\text{rank}_W M) \cdot \text{tr}_{F/\mathbb{Q}}(a), \quad \forall a \in O_F.$$

*Then there is a supersingular principally polarized  $O_F$ -abelian variety  $\underline{A} = (A, \lambda, \iota)$  over  $k$  whose Dieudonné module  $M(\underline{A})$  is isomorphic to  $\underline{M}$ .*

(2) *Assume that  $p$  is totally ramified in  $F$ . Then for any supersingular Dieudonné  $\mathcal{O}$ -module  $\underline{M} = (M, \iota_M)$  over  $k$  of  $W$ -rank  $2[F : \mathbb{Q}]$ , there is a principally polarized  $O_F$ -abelian variety  $\underline{A} = (A, \lambda, \iota)$  over  $k$  such that the Dieudonné  $\mathcal{O}$ -module  $M(\underline{A}, \iota)$  is isomorphic to  $\underline{M}$ .*

PROOF. (1) By [13, Theorem 1.1], there is a (prime-to- $p$  degree) polarized  $O_F$ -abelian variety  $\underline{A} = (A, \lambda, \iota)$  such that  $M(\underline{A}) \simeq \underline{M}$ . We can choose a self-dual  $(O_F \otimes \mathbb{Z}_\ell)$ -lattice  $L_\ell$  in  $V_\ell(A)$  with respect to  $e_{\lambda, \ell}$  for each prime  $\ell \neq p$  with  $L_\ell = T_\ell(A)$  for almost all  $\ell$ . The proof of the existence of such a lattice  $L_\ell$  is elementary and left to the reader. Then there is an  $O_F$ -abelian variety  $(A', \iota')$  and a prime-to- $p$  degree  $O_F$ -linear quasi-isogeny  $\varphi' : (A', \iota') \rightarrow (A, \iota)$  such that  $\varphi'_*(T_\ell(A')) = L_\ell$  for all  $\ell \neq p$ . Then the pull-back  $\lambda' := \varphi'^*\lambda$  by  $\varphi$  is a principal polarization as  $L_\ell$  is self-dual. The object  $(A', \lambda', \iota')$  is a desired one.

(2) Since there is only one prime of  $O_F$  over  $p$ , the condition  $(*)$  is satisfied. By [10, Proposition 2.8], the Dieudonné  $\mathcal{O}$ -module  $\underline{M}$  admits a separable  $\mathcal{O}$ -linear quasi-polarization, noting that an equivalent condition (5) of loc. cit. is satisfied when  $p$  is totally ramified. Then the statement follows from (1). ■

Now we return to our case  $F = \mathbb{Q}(\sqrt{p})$ , where  $\mathcal{O} = O_F \otimes \mathbb{Z}_p = \mathbb{Z}_p[\sqrt{p}]$ . The prime  $p$  is ramified in  $F$  with ramification index  $e = 2$ . Clearly any member  $A$  in  $\Lambda(\pi)^{\max}$  is a superspecial abelian surface over  $\mathbb{F}_p$ . The Dieudonné module  $M = M(A)$  of  $A$  is a rank 4 free  $\mathbb{Z}_p$ -module together with a  $\mathbb{Z}_p$ -linear action by  $O_F$ . Therefore,  $M \simeq \mathcal{O}^2$  on which both the Frobenius  $\mathcal{F}$  and the Verschiebung  $\mathcal{V}$  operate by  $\sqrt{p}$ . From this the Lie algebra  $\text{Lie}(A) = M/\mathcal{V}M$  of  $A$  is isomorphic to  $\mathbb{F}_p \oplus \mathbb{F}_p$  as an  $(O_F/p)$ -module. In other words,  $A$  has Lie type  $(1, 1)$  in the terminology of [10, Section 1]. Therefore, the injective map  $\Phi_\pi : \Lambda(\pi)^{\max} \rightarrow \mathbf{B}_{2, O_F}$  factors through the subset  $\mathbf{S} \subset \mathbf{B}_{2, O_F}$  of superspecial abelian  $O_F$ -surfaces of Lie type  $(1, 1)$ .

We first claim that the induced map

$$(5.7) \quad \Phi_\pi : \Lambda(\pi)^{\max} \rightarrow \mathbf{S}$$

is bijective. Fix a member  $A_0 \in \Lambda(\pi)^{\max}$ . By Waterhouse [9, Theorem 6.2], there is a natural bijection between the set  $\Lambda(\pi)^{\max}$  and the set  $\text{Cl}(\text{End}(A_0))$  of right ideal classes. Since the map  $\Phi_\pi$  is injective, it suffices to show that  $\mathbf{S}$  has the same cardinality as  $\text{Cl}(\text{End}(A_0))$ . Note that the isomorphism classes of (unpolarized) *superspecial* Dieudonné  $\mathcal{O}$ -modules are uniquely determined by their Lie types [12, Lemma 3.1]. It follows that the Dieudonné modules and Tate modules of any two members in  $\mathbf{S}$  are mutually isomorphic (compatible with the actions of  $O_F$ ). By (the unpolarized variant of) [12, Theorem 2.1], there is a natural bijection  $\mathbf{S} \simeq \text{Cl}(\text{End}_{O_F}(\overline{A}_0))$ . Since we have  $\text{End}(A_0) = \text{End}_{O_F}(\overline{A}_0)$ , our claim is proved.

Let  $\mathbf{S}_1 \subset \mathbf{B}_{2,O_F,1}$  be the subset consisting of objects  $(A, \lambda, \iota)$  so that the underlying abelian  $O_F$ -surface  $(A, \iota)$  belongs to  $\mathbf{S}$ . Proposition 5.3 implies that  $\mathbf{S}_1$  is nonempty. Consider the commutative diagram

$$(5.8) \quad \begin{array}{ccc} \Lambda(\pi)_1^{\max} & \xrightarrow{\Phi_\pi} & \mathbf{S}_1 \\ f_\Lambda \downarrow & & f_{\mathbf{S}} \downarrow \\ \Lambda(\pi)^{\max} & \xrightarrow[\simeq]{\Phi_\pi} & \mathbf{S} \end{array}$$

Note that a member  $A$  in  $\Lambda(\pi)^{\max}$  admits a principal polarization if and only if  $\Phi_\pi(A) = (\overline{A}, \iota)$  admits a principal  $O_F$ -linear polarization. Moreover, the equivalence classes of principal polarizations on  $A$  are in bijection with the equivalence classes of principal  $O_F$ -linear polarizations on  $(\overline{A}, \iota)$ . It follows that the diagram (5.8) is cartesian, which particularly implies that the map  $\Phi_\pi : \Lambda(\pi)_1^{\max} \simeq \mathbf{S}_1$  is an isomorphism. Thus, we have proved  $\text{Mass}(\Lambda(\pi)_1^{\max}) = \text{Mass}(\mathbf{S}_1)$ .

Now we use the mass formula for  $\text{Mass}(\mathbf{S}_1)$  [12, Theorem 3.7]\*

$$(5.9) \quad \text{Mass}(\Lambda(\pi)_1^{\max}) = \text{Mass}(\mathbf{S}_1) = \frac{1}{4} \zeta_F(-1);$$

this proves Theorem 5.2.

**5.3. Fibers of the map  $f_{\mathbf{S}}$ .** We describe the fibers of the map  $f_{\mathbf{S}}$  in (5.8). Suppose  $(A, \lambda_0, \iota)$  is a member in  $\mathbf{S}_1$ . Put  $D := \text{End}_{O_F}^0(A)$  and  $O_D := \text{End}_{O_F}(A)$ . Then  $D$  is the quaternion  $F$ -algebra ramified only at the two real places of  $F$  and  $O_D$  is a maximal order. Note that the canonical involution  $'$  is the unique positive involution on  $D$ . Therefore the Rosati involution induced by any  $O_F$ -linear polarization must be  $'$ . Suppose  $\lambda$  is another  $O_F$ -linear principal polarization, then  $\lambda = \lambda_0 a$  for some totally positive symmetric element  $a \in O_D^\times$ , so  $a \in O_{F,+}^\times$ , the set of totally positive units in  $O_F$ . Suppose  $b \in \text{Aut}_{O_F}(A)$  is an  $O_F$ -linear automorphism. Then the pull-back

$$b^*(\lambda_0 a) = b^t \lambda_0 a b = \lambda_0 \lambda_0^{-1} b^t \lambda_0 b a = \lambda_0 (b' b) a.$$

Therefore, the set of equivalence classes of principal  $O_F$ -linear polarizations on  $(A, \iota)$  is in bijection with the set  $O_{F,+}^\times / \text{Nr}(O_D^\times)$ , where  $\text{Nr} : O_D \rightarrow O_F$  is the

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\*There is an error in the computation of the mass formula there. The error occurs in Lemma 3.4 of loc. cit., where the unramified quadratic order  $O_{\mathbf{F}_p}$  of  $O_{\mathbf{F}_p}$  cannot be written as  $O_{\mathbf{F}_p}[\sqrt{c}]$  when  $p = 2$  as stated. As a result, the order  $A_\epsilon$  when  $\epsilon = 0$  as in Lemma 3.4 should be maximal, and the term  $o_p$  should be always one in that paper, particularly in the formulas of Theorems 4.4 and 4.5.

reduced norm. In other words, we obtain an isomorphism

$$(5.10) \quad f_{\mathbf{S}}^{-1}(A, \iota) \simeq O_{F,+}^{\times} / \mathrm{Nr}(O_D^{\times}).$$

As  $\mathrm{Nr}(O_D^{\times}) \supset (O_F^{\times})^2$ , the group  $O_{F,+}^{\times} / \mathrm{Nr}(O_D^{\times})$  is a homomorphism image of  $O_{F,+}^{\times} / (O_F^{\times})^2$ . The latter group has 1 or 2 elements according as the fundamental unit  $\epsilon$  of  $F$  has norm  $-1$  or not. Therefore, if  $N(\epsilon) = -1$ , then  $f_{\mathbf{S}}^{-1}(A, \iota)$  has one element. Otherwise, the fiber  $f_{\mathbf{S}}^{-1}(A, \iota)$  has at most two elements.

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